ALMOST MINIMIZING YANG–MILLS FIELDS: LOG-EPIPERIMETRIC INEQUALITY, NON-CONCENTRATION, AND UNIQUENESS OF TANGENTS

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ABSTRACT. We establish a direct log-epiperimetric inequality for Yang–Mills fields in arbitrary dimension and we leverage on it to prove uniqueness of tangent cones with isolated singularity for energy minimizing Yang–Mills fields and ω-ASD connections (where ω is not necessarily closed) satisfying some suitable regularity assumptions. En route to this we establish a Luckhaus type lemma for Yang–Mills connections to exclude curvature concentration along blow-up sequences.

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CONTENTS

1. INTRODUCTION

1.1. **The general setting: Yang–Mills fields.** The aim of this article is to understand the behaviour of Yang–Mills connections at their singular points, and prove uniqueness of the corresponding tangent cones whenever they happen to satisfy certain structural properties. This fits broadly into a line of investigation aiming at understanding the regularity of extrema of geometric variational problems, and solutions of partial differential equations of geometric type.

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Let us recall the framework we will be working in. Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$ and (N,h) a smooth n-dimensional Riemannian manifold with $n \geq 2$, possibly with smooth boundary ∂N. The *Yang–Mills functional* on a principal G-bundle P over N is given by

$$\mathrm{YM}_N(A) := \int_N |F_A|^2 \, d\,\mathrm{vol}_h \qquad \forall\, A \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g}_P), \tag{1.1}$$

where $F_A := dA + A \wedge A \in L^2(N, \wedge^2 T^*N \otimes \mathfrak{g}_P)$ is the curvature of the principal G-connection A, and the norm $|F_A|$ is computed with respect to the Ad-invariant inner product on the *adjoint bundle* $\mathfrak{g}_P$[1] over N. Critical points of YM_N are called *Yang–Mills fields* or *Yang–Mills connections* on P.[2] Geometrically, the Yang–Mills energy measures how far a certain connection is from being flat in the L^2-sense. From an analytic perspective, YM_N is a conformally invariant[3] lagrangian in its *critical* dimension $n = 4$, whose properties make the associated variational problems rich and challenging. For instance, the functional is *gauge invariant* in the following precise sense: for every *local gauge transformation* $g \in (W^{2,2} \cap W^{1,4})(N, G_P)$[4] and for every connection A, the *gauge transformed* connection

$$A^g := g^{-1}dg + g^{-1}Ag \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g}_P)$$

satisfies

$$\mathrm{YM}_N(A^g) = \mathrm{YM}_N(A).$$

The presence of such a large invariance group for YM_N leads to several remarkable analytical consequences. For example, the Euler–Lagrange equations, called *Yang–Mills equations*, and the stability operator associated with YM_N are *not elliptic*, at least until a proper *relative Coulomb gauge* is chosen, cf. Remark 3.3. Besides, the nonlinearity $A \wedge A$ appearing in these equations is unwieldy, as it drastically breaks the coercivity of the functional leading to concentration-compactness phenomena in dimension $n \geq 4$.[5]

The Yang–Mills functional has played a major role in the understanding of the differential geometry of 4-manifolds. Let us recall a few key examples, without aiming for completeness (see e.g. [DK90] or [FU91] for a detailed discussion of the subject). Donaldson proved his

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[1]The adjoint bundle $\mathfrak{g}_P$ is the vector bundle $\tilde{\pi} : \mathfrak{g}_P \to N$ over N whose total space is given by

$$\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g} = \frac{P \times \mathfrak{g}}{\sim_{\mathrm{Ad}}}$$

where

$$(p_1, v_1) \sim_{\mathrm{Ad}} (p_2, v_2) \quad \Leftrightarrow \quad \exists\, g \in G : \; p_2 = p_1 g, \; v_2 = g^{-1} v_1 g$$

and whose projection on the base manifold N is defined to be $\tilde{\pi}([(p,v)]) := \pi(p)$ for every $[(p,v)] \in \mathfrak{g}_P$.

[2]If $P := N \times G$ is the trivial principal G-bundle over N, for short we said that critical points of YM_N are Yang–Mills fields on N.

[3]Meaning invariant with respect to rescalings in the domain.

[4]Here G_P stands for the *conjugated bundle*, i.e. the G-bundle over N whose total space is given by

$$G_P := P \times_c G = \frac{P \times G}{\sim_c}$$

where

$$(p_1, h_1) \sim_c (p_2, h_2) \quad \Leftrightarrow \quad \exists\, g \in G : \; p_2 = p_1 g, \; h_2 = g^{-1} h_1 g$$

and whose projection on the base manifold N is defined to be $\hat{\pi}([(p,h)]) := \pi(p)$ for every $[(p,h)] \in G_P$.

[5]See e.g. [Uhl82a] and [Tia00], where the authors studied the concentration-compactness phenomena for the Yang–Mills functional respectively in critical and supercritical dimension.

celebrated result on the existence of non-smoothable topological 4-manifolds by studying properties of the the moduli space of instantons (special symmetric solutions of the Yang–Mills equations) over such manifolds (see [Don83]). Notably, these manifolds had already been constructed by Freedman a year earlier in his solution to the topological 4-dimensional Poincaré conjecture [Fre82]. A second remarkable application of gauge theory in this context is the proof of existence of exotic differentiable structures on $\mathbb{R}^4$. The existence of at least one "fake" $\mathbb{R}^4$ stems largely from the aforementioned works of Donaldson and Freedman, with Gompf later providing an explicit construction and showing the existence of at least three such exotic structures [Gom83]. Finally, Taubes in [Tau87] achieved the sharpest result in this sense, showing the existence of uncountably many fake $\mathbb{R}^4$'s by means of gauge theoretic methods.

Given the effective applications of YM_N in four dimensions, it is natural to investigate its behaviour in higher dimensions, i.e. in the *supercritical* regime. Notably, a research program along these lines was proposed by Donaldson and Thomas in [DT98]. Unfortunately, the analysis of the Yang–Mills lagrangian becomes more challenging in dimension greater than four and we are confronted with the study of *singular* solutions, as they naturally arise in this wilder framework. As in other geometric variational problems, it is useful to consider tangent cones, *tangent connections* in this case. These are weak limits of rescalings of the original connection, capturing the local behaviour around a given point. As these limiting objects are supposed to model the behaviour around a (singular) point, understanding whether they are unique is an important issue. A priori, at a given singular point the connection may asymptotically approach one cone at certain scales, and a different cone at others.

A method to prove uniqueness of tangents with *isolated singularity* was pioneered by Simon in [Sim83a] for stationary varifolds, and harmonic maps. The idea is to prove an infinite dimensional version of the classical Łojasiewicz gradient inequality for analytic functions in Euclidean space, cf. Lemma 4.1. In gauge theory, this method was first introduced by Morgan, Mrowka, and Ruberman in their work [MMR94] on the Chern–Simons functional, while Råde [Rd92] later adapted Simon's proof in dimensions 2 and 3 for the Yang–Mills functional. It was only later that Yang [Yan03] used Simon's method in any dimension to prove that, under strong curvature bounds, tangent cones at isolated singularities are unique. In this article we follow a different approach to this old question, and we remove the assumption on the curvature, thus generalising Yang's result [Yan03]. We also refer the reader to more recent works of Feehan, partly in collaboration with Maridakis, establishing various gradient Łojasiewicz inequalities [Fee16, Fee22, FM20b, FM20a].

Łojasiewicz–Simon inequality type arguments are not directly applicable if the singularity of the cone is not isolated, as they usually require the cone to have a smooth link. In the case of pseudoholomorphic maps and semicalibrated currents, slicing techniques proved to be extremely effective (see e.g. [PR10], [RT04], [CR23]). In the particular case of integral $p-p$ cycles, we also mention the work [Bel14] by Bellettini, in which the author develops the so called *algebraic blow-up method*. We currently conjecture that these type of techniques could be the key to tackle the uniqueness of tangents for general ω-ASD connections[6] on general almost complex manifolds. Note that in the setting of stationary integral 1-varifolds, uniqueness of tangents is completely settled by [AA81]. We refer the reader to the surveys [DL22, Wic14] for further details on the uniqueness of tangent cones problem in the setting of minimal submanifolds.

[6]See Definition 2.6.

1.2. **Statement of the main results.** Since all our results are local, from now on we will work on the trivial principal G-bundle $P = \Omega \times G$ over an open set $\Omega \subset \mathbb{R}^n$. This corresponds to taking $\mathfrak{g}_P = \mathfrak{g}$ and $G_P = G$ in the previous definitions and notation. In comparison with previous literature on the subject, we make minimal regularity assumptions on the class of connections under consideration. Secondly, we go beyond the Kähler setting, in which the richer complex structure, and in particular the existence of local holomorphic coordinates, simplifies the analysis and allows for finer considerations. See below for further details on this aspect. Besides, our proof is purely variational, and does not exploit any underlying PDE, thus allowing us to treat in a unified and more systematic way a larger class of extrema, i.e. almost minimizers. Our first main theorem is the following.

Theorem 1.1 (Uniqueness of tangents). *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 5$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that the following facts hold.*

(1) $A \in (W^{1,2} \cap L^4)(\Omega, T^*\Omega \otimes \mathfrak{g})$ is either an ω-ASD connection on Ω or a YM*-energy minimizer*[7] *such that $\mathscr{H}^{n-4}(\mathrm{Sing}(A) \cap K) < +\infty$ for every compact set $K \subset \Omega$.*[8]

(2) $y \in \mathrm{Sing}(A)$ and φ is a tangent cone for A at y such that $\mathrm{Sing}(\varphi) = \{0\}$ and $F_{A_{y,\rho_i}} \to F_\varphi$ strongly in L^2 (modulo gauge transformations) along some sequence of rescalings $\rho_i \to 0$ as $i \to +\infty$.

Then, φ is the unique tangent cone to A at y (modulo gauge transformations). Moreover, the decay is logarithmic, i.e. there exist $\alpha > 0$ and constants $C_k > 0$ for every $k \in \mathbb{N}$ such that

$$|\tau_{y,\rho}^* A - \varphi|_{C^k(\mathbb{S}^{n-1})} \leq C_k |\log(\rho)|^{-\alpha},$$

where $\tau_{y,\rho}(x) = \rho x + y$ is the rescaling of factor $\rho > 0$ centered at y. Moreover, if we assume φ to be integrable, the rate of convergence improves to

$$|\tau_{y,\rho}^* A - \varphi|_{C^k(\mathbb{S}^{n-1})} \leq C_k \rho^\alpha.$$

Remark 1.2. Throughout this work, ω we will never assume that ω is closed. This entails that our ω-ASD connections are just "almost" YM-energy minimizers[9] in general.

We briefly comment the main hypotheses in the above theorem.

- $\mathscr{H}^{n-4}(\mathrm{Sing}(A) \cap K) < +\infty$ for every compact set $K \subset \Omega$. This in particular implies that A is an *admissible connection* in the sense of [Tia00]. Thus, by [TT04], an ε-regularity statement is available for A. In fact, if A belongs to any class with such property, for example the *strongly approximable connections* introduced in [MR03], then we can immediately prove Theorem 1.1 following the argument in [Sim12, Section 3.15]. We also point out upcoming work of the first author with Rivière in which an ε-regularity statement is obtained for *weak L^2-connections* in dimension 5, first introduced by [PR17] as a suitable variational framework for the Yang–Mills lagrangian in the first supercritical dimension.
- The tangent cone φ has an isolated singularity at the origin. This is the usual hypothesis appearing in [Sim83a, ESV19, ESV20] and beyond which it is incredibly difficult to

[7] See Definition 2.5.

[8] Here and in what follows, we let

$$\mathrm{Reg}(A) := \{x \in \Omega : A \in C^\infty(B_\rho(x), T^* B_\rho(x) \otimes \mathfrak{g}) \text{ for some } \rho > 0\}$$

and $\mathrm{Sing}(A) = \Omega \setminus \mathrm{Reg}(A)$. Moreover, $\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^n$.

[9] See Definition 2.5.

go. In the setting of mean curvature flow, Colding and Minicozzi have been able to deal with (generic) *cylindrical* singularities, see [CM15]. We also refer the reader to work of Székelyhidi on cylindrical tangent cones, [Szé20], and to earlier work of Simon [Sim93, Sim94].

- Non-concentration of the measures, i.e. emptyness of the bubbling locus. This is another additional hypothesis due to the potential bubbling of sequences of Yang–Mills connections. In particular, the curvatures of the rescaled connections $\tau_{r,y}^* A$ could exhibit a concentration set larger than the actual singular set of φ. This possibily is ruled out in the context of harmonic maps. Indeed, when considering a sequence of *energy minimizing* harmonic maps weakly converging in $W^{1,2}$, we know that the convergence is in fact strong [SU82], and the limit is energy minimizing as well [HL87, Luc88].

We overcome this last difficulty by proving the a Luckhaus type lemma for Yang–Mills connections, currently not available in literature, thus allowing us to completely exclude curvature concentration in dimension 5. We can also rule out this phenomenon in higher dimensions, upon requiring the connection to be more regular (cf. Lemma 6.2). See [Wal19] for related issues of concentration for the parabolic Yang–Mills flow.

Theorem 1.3 (Uniqueness of tangents excluding concentration). *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 5$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that the following facts hold.*

(1) $A \in (W^{1,\frac{n-1}{2}} \cap L^{n-1})(\Omega, T^*\Omega \otimes \mathfrak{g})$ is either an ω-ASD connection on Ω or a YM*-energy minimizer such that $\mathscr{H}^{n-4}(\operatorname{Sing}(A) \cap K) < +\infty$ for every compact set $K \subset \Omega$.*
(2) $y \in \operatorname{Sing}(A)$ and φ is a tangent cone for A at y such that $\operatorname{Sing}(\varphi) = \{0\}$.

Then, φ is the unique tangent cone to A at y (modulo gauge transformations). Moreover, the decay is logarithmic, i.e. there exist $\alpha > 0$ and constants $C_k > 0$ for every $k \in \mathbb{N}$ such that

$$|\tau_{y,\rho}^* A - \varphi|_{C^k(\mathbb{S}^{n-1})} \leq C_k |\log(\rho)|^{-\alpha},$$

where $\tau_{y,\rho}(x) = \rho x + y$ is the rescaling of factor $\rho > 0$ centered at y. Moreover, if we assume φ to be integrable, the rate of convergence improves to

$$|\tau_{y,\rho}^* A - \varphi|_{C^k(\mathbb{S}^{n-1})} \leq C_k \rho^{\alpha}.$$

As a corollary of the above theorem, we obtain the following sharp result in dimension 5 for connections in the natural class $W^{1,2} \cap L^4$ appearing in Theorem 1.1.

Corollary 1.4. *When $n = 5$, the class of connections of Theorem 1.3 reduces to the natural one $(W^{1,2} \cap L^4)(\Omega, T^*\Omega \otimes \mathfrak{g})$, thus implying that we have curvature non-concentration in dimension 5 in the setting of Theorem 1.1.*

Remark 1.5. Although not explicitly stated, it follows from the two-step degeneration theory developed in [CS20a, CS20b, CS21a, CS21b] that the logarithmic decay is the best possible one. Indeed, when the algebraic tangent cone and the analytic one coincide, the rate is polynomial. In particular, considering examples where the two types of cones are different would give the desired logarithmic decay. We believe that it would still be interesting to construct such examples more explicitly, as done for minimal surfaces and harmonic maps in [AS88, Section 5], see also [GW89]. We also refer the reader to [CM14] for a similar issue in the setting of Einstein manifolds, and to [SZ23] for the one of singular Kähler–Einstein metrics.

In the special case in which the underlying manifold is Kähler (X, ω) and the vector bundle (E, H) is Hermitian, more refined structural results have been obtained. For instance, the special class of *admissible Hermitian Yang–Mills* connections has received a lot of interest in recent years, see [CS20b] for the definition. For instance, in loc. cit. the authors are able to relate tangents cones of admissible Hermitian Yang–Mills connections at an isolated singularity to the complex algebraic geometry of the underlying reflexive sheaf (modulo an extra hypothesis on the Harder–Narasimhan–Seshadri filtration). See also [JSEW18] for a proof in the case in which the vector bundle over $\mathbb{CP}^{n-1}$ is a direct sum of polystable bundles. A crucial tool in [CS20b] is the notion of *algebraic tangent cone* already mentioned in Remark 1.5, and how it relates to the notion of (analytic) tangent cone that we introduced earlier in this introduction. Very loosely speaking this is a torsion-free sheaf over an exceptional divisor satisfying some extra requirements, and it carries information on the underlying singularities. In [CS21a] Chen and Sun were able to establish an algebro-geometric characterization of the bubbling set, always in the setting of isolated singularities (see also [CS20a]). They then go on and conclude with [CS21b] by fully resolving this dichotomy between different notions of cones. We note that a key analytic tool of [CS20b], similar to our log-epiperimetric inequality, is a convexity result in the form of a *three circle lemma*. We conclude this subsection by motivating the study of this class of connections.

(i) From the complex geometric point of view, Bando and Siu [BS94] proved that polystable reflexive sheaves over a compact Kähler manifold always admit an admissible Hermitian Yang–Mills connection. This generalised the Donaldson–Uhlenbeck–Yau theorem for holomorphic vector bundles, meaning that these objects are relevant in algebraic geometry.
(ii) From the gauge theoretic perspective, by [Nak88, Tia00] admissible Hermitian Yang–Mills connections naturally arise in the compactification of the moduli space of smooth ones. Therefore, they play an important role in understanding the structure of the moduli space in gauge theory over higher dimensional Kähler manifolds.
(iii) When doing gauge theory over G_2 manifolds, it is expected that singularities of this special class of connections in dimension three model singularities of G_2 instantons. We refer the reader to [JW18, SEW15] for further details on this.

The main ingredient in the proof of Theorem 1.1, Theorem 1.3, and Corollary 1.4 is a new log-epiperimetric inequality for the Yang–Mills lagrangian, a quantitative estimate on the suboptimality of the homogeneous extension that we now describe, cf. Theorem 1.6 for the main estimate.

1.3. **Epiperimetric Inequalities.** Direct epiperimetric inequalities were introduced in seminal work of Reifenberg [Rei64a] in the context of minimal surfaces with the aim of proving that solutions of the Plateau problem, as posed by the author, were analytic [Rei64b]. White later exploited this idea in [Whi83] to establish uniqueness of tangent cones for two-dimensional area-minimizing integral currents without boundary in $\mathbb{R}^n$. This result was then extended by Rivière in [Riv04] where the author introduced the notion of *lower* epiperimetric inequality, and proved the corresponding uniqueness of tangents. We note that as a consequence of this inequality, Rivière exhibited and investigated the phenomenon of *splitting before tilting*, pivotal for the proof of regularity of $1-1$ integral currents joint with Tian [RT09], and crucial for later developments on the regularity theory of area-minimizing and semicalibrated currents by De Lellis, Spadaro, and Spolaor, see [DLS16, DLSS17a, Spo19]. As part of a program to investigate the regularity of two-dimensional almost minimal currents, these last three authors recently established in [DLSS17b] an epiperimetric inequality in this setting, thus generalising the aforementioned work of White.

The introduction of direct epiperimetric inequalities to the framework of free boundary problems is due to Spolaor and Velichkov in [SV19]. Subsequently, these last two authors, together with Colombo, proved a novel *logarithmic epiperimetric inequality* in the context of obstacle-type problems [CSV18, CSV20a]. In a nutshell, this is a quantitative estimate on the optimality of the homogeneous extension which gives a logarithmic decay to the blow-up. The additional terms in the inequality are due to the possible presence of elements in the kernel of a suitable linearized operator. In [ESV20], always Spolaor and Velichkov, together with Engelstein, developed a new approach for proving this inequality based on reducing it to a quantitative estimate for a functional defined on the unit sphere, and studying the corresponding gradient flow. This was done for the Alt-Caffarelli functional, but the new perspective found fruitful applications to the study of multiplicity-one stationary cones with isolated singularities [ESV19], bearing with them new ε-regularity results for almost minimizers. See also [SV21] and [ESV24].

All of the results mentioned above are direct in the sense that they are based on an explicit construction of a competitor. This is usually more adapted to establishing decay estimates around singular points. However, a large class of epiperimetric inequalities are proven by contradiction. These are based on linearization techniques and the contradiction arguments appearing in the literature apply to regular points or singular points with additional structural hypothesis. In the setting of minimal submanifolds we mention works of Taylor on area-minimizing flat chains modulo 3 and $(\mathbf{M}, \varepsilon, \delta)$-minimizers [Tay73, Tay76a, Tay76b], while for free boundary problems the first instance of an epiperimetric inequality is due to Taylor in [Tay77]. Later on, Weiss in [Wei99] introduced an epiperimetric inequality in the setting of the classical obstacle problem at flat singular points and along the top stratum of the singular set. On the other hand, for the thin obstacle problem, we refer the reader to [FS16, GPSVG16]. Eventually, for free boundary problems for harmonic measures we mention work of Badger, Engelstein, and Toro [BET20] whose great novelty is to apply an epiperimetric inequality for functions that do not minimize any energy. Our main contribution to this line of investigation in the setting of the Yang–Mills Lagrangian is the following.

Theorem 1.6 (Log-epiperimetric inequality). *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$ and let $n \geq 5$. Let $A_0 \in C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ be a smooth Yang–Mills connection and define the 1-form $\tilde{A}_0 \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ to be its 0-homogeneous extension inside $\mathbb{B}^n$, given by*

$$\tilde{A}_0 := \left(\frac{\cdot}{|\cdot|}\right)^* A_0.$$

There exist constant $\varepsilon, \delta > 0$, and $\gamma \in [0, 1)$ depending on the dimension and A_0 such that the following holds. If $A \in C^{2,\alpha}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ is such that

$$\|A - A_0\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} < \delta$$

then there exists $\hat{A} \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ such that $\iota_{\mathbb{S}^{n-1}}^* \hat{A} = A$ and

$$\mathscr{Y}_{\mathbb{B}^n}(\hat{A}\,; \tilde{A}_0) \leq \big(1 - \varepsilon |\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}\,; \tilde{A}_0)|^\gamma\big) \mathscr{Y}_{\mathbb{B}^n}(\tilde{A}\,; \tilde{A}_0), \tag{1.2}$$

where $\tilde{A} \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ is the 0-homogeneous extension of A inside $\mathbb{B}^n$. Furthermore, if the kernel of the second variation is integrable, we can take $\gamma = 0$.

Remark 1.7. We expect that modifying the proof of Theorem 1.6 as in [ESV24] to prove a symmetric log-epiperimetric inequality, one can prove uniqueness of tangent cones at infinity for sequences of connections. In particular, one would appeal to the decay-growth Theorem à la Edelen-Spolaor-Velichkov, cf. [ESV24, Theorem 2.2].

1.4. **Ideas of the proofs and structure of the article.** The proof of Theorem 1.6 follows the strategy outlined in [ESV19, ESV20]. In particular, it relies on a careful construction of a competitor function with energy smaller than the one of the 0-homogeneous extension of A. The starting point is a slicing lemma to write the energy discrepancy $\mathscr{Y}_{\mathbb{B}^n}(\cdot\,;\cdot)$ in Theorem 1.6 in a more convenient form, cf. Lemma 3.2. This is done in Section 3, where we also recall the Lyapunov–Schmidt reduction adapted to our setting, cf. Lemma 3.4. An additional difficulty that we have to overcome here is the kernel of the second variation being infinite dimensional. This is due to the gauge invariance of the Yang–Mills lagrangian. A similar issue was faced in [CM14], where the authors have to mode out the diffeomorphism invariance of their functional $\mathcal{R}$ (they do so by the Ebin–Palais slice theorem). Similarly, Simon worked with normal graphs to avoid this issue [Sim83a], while Yang [Yan03] introduced a form of transverse gauge. We resolve this by simply working in a suitable relative Coulomb gauge.

This different way of writing the energy discrepancy suggests that we can construct the competitor by flowing inwards the components of the trace A in the directions that decrease the energy at second order around the critical point A_0 with respect to which we want to compute the energy discrepancy. To choose the appropriate directions of the flow we turn to the second variation at A_0. We write it as a linear elliptic operator with compact resolvent, thus implying that it has a finite dimensional kernel. Consequently, we can decompose the datum A as the sum of the projections on the kernel, the positive, and the negative eigenvalues, i.e. the index. As A_0 is Yang–Mills on the sphere $\mathbb{S}^{n-1}$, positive directions will increase the energy to second order, while negative directions will decrease it. Whence, we want to move A_0 towards zero in the former, while keeping the latter fixed. To deal with the kernel we resort to a *finite dimensional Yang–Mills flow*. To make the estimate on $\mathscr{Y}_{\mathbb{B}^n}(\cdot,\cdot)$ more quantitative, we appeal to the finite dimensional Łojasiewicz inequality, cf. Lemma 4.1, which is ultimately responsible for the error term appearing in (1.2) and is the reason why our inequality is (log)-epiperimetric instead of just epiperimetric. The proof of Theorem 1.6 just outlined appears Section 4. In the integrable case, i.e. when the projection of A_0 on the kernel of the second variation vanishes, the proof simplifies significantly, see Subsection 4.1. Note that the logarithmic error term is unavoidable when considering nonintegrable singularities, and that in the setting of stationary varifolds, there is also a more restrictive notion of *integrable through rotations*, see [ESV19, Remark 1.3]. Beyond its remarkable flexibility, one of the fundamental insights of this strategy is that it draws a precise relationship between the kernel of the second variation and the logarithmic decay term in the epiperimetric inequality.

Remark 1.8. If one had a Łojasiewicz–Simon inequality for Sobolev connections, the proof of Theorem 1.6 could be simplified by simply flowing inwards the full trace, and not just its projection onto the kernel. This is explained in [CSV20b, Proposition 3.1]. In particular, from a Łojasiewicz–Simon inequality descends a log-epiperimetric one. Unfortunately, as already mentioned above, for Sobolev connections we only have the former in dimensions 2, 3, 4. It would be interesting to bridge this gap.

The proof of Theorem 1.1 is inspired by work of Simon [Sim83a] and it appears in Section 5. We exploit the log-epiperimetric inequality to establish a bound for the energy density

$\Theta(\rho, y; A) - \mathrm{YM}_{\mathbb{B}^n}(\varphi)$ at all dyadic scales, which can then be converted to a bound at all scales. Uniqueness of the tangent map then follows by a standard Dini-type estimate that we include in Appendix A. We deal with potential concentration phenomena in Section 6.

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2. Preliminaries on the Yang–Mills functional

In this section we collect the basic definitions and properties of the Yang–Mills lagrangian. In particular, we prove that this functional is analytic. We also introduce the class of almost minimizers of the Yang–Mills energy, and define ω-anti-self-dual connections. We then prove that the latter belong to the former. We prove an almost monotonicity formula resembling the one for semicalibrated currents. We conclude by explaining the phenomenon of concentration appearing in Theorem 1.1.

2.1. The Yang–Mills lagrangian and Yang–Mills connections.

Definition 2.1 (The Yang–Mills functional). Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$ and let $n \geq 2$ and let (N, h) be a smooth n-dimensional Riemannian manifold, possibly with smooth boundary ∂N.
The *Yang–Mills functional* $\mathrm{YM}_N : (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g}) \to [0, +\infty)$ on the trivial bundle over N is given by

$$\mathrm{YM}_N(A) := \int_N |F_A|^2 \, d\,\mathrm{vol}_h \qquad \forall A \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g}),$$

where

$$F_A := dA + A \wedge A \in L^2(N, \wedge^2 T^*N \otimes \mathfrak{g}).$$

Given any open subset $U \subset N$, for every $A \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})$ we let

$$\mathrm{YM}_N(A\,; U) := \int_U |F_A|^2 \, d\,\mathrm{vol}_h$$

be the *Yang–Mills energy of A localized in U*.

Definition 2.2 (Yang–Mills connections). Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 2$ and let (N, h) be a smooth n-dimensional Riemannian manifold, possibly with smooth boundary ∂N.
A *Yang–Mills* connection on the trivial bundle over N is a critical point of YM_N.

Definition 2.3 (YM-energy discrepancy). Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 2$ and let (N, h) be a smooth n-dimensional Riemannian manifold, possibly with smooth boundary ∂N. Let $A_0 \in (W^{1,2} \cap L^4)(N, \wedge^1 T^*N \otimes \mathfrak{g})$. The functional $\mathscr{Y}_N(\,\cdot\,; A_0)$ given by

$$\mathscr{Y}_N(A\,; A_0) := \mathrm{YM}_N(A) - \mathrm{YM}_N(A_0) \qquad \forall\, A \in (W^{1,2} \cap L^4)(N, \wedge^1 T^*N \otimes \mathfrak{g})$$

is called YM-*energy discrepancy with respect to* A_0 on N.

Proposition 2.4. *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 2$ and let (N,h) be a smooth n-dimensional Riemannian manifold, possibly with smooth boundary ∂N. Given any $A_0 \in (W^{1,2} \cap L^4)(N, \wedge^1 T^*N \otimes \mathfrak{g})$, the following facts hold.*

(i) The functional YM_N is a quartic functional on $(W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})$, i.e. given any $A \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})$ for every $k = 0,1,2,3,4$ there exists a k-linear and bounded operator

$$\nabla^k \mathrm{YM}_N(A) : (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})^k \to \mathbb{R}$$

such that

$$\mathrm{YM}_N(A + \varphi) = \sum_{k=0}^{4} \frac{\nabla^k \mathrm{YM}_N(A)}{k!} [\underbrace{\varphi, ..., \varphi}_{k \text{ times}}] \qquad \forall \, \varphi \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g}).$$

(ii) The functional $\mathscr{Y}_M(\,\cdot\,; A_0)$ is real-analytic on $(W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})$ and its first and second Fréchet differentials are given by[10]

$$\nabla \mathscr{Y}_N(A\,; A_0)[\varphi] = 2 \int_N \langle F_A, d_A \varphi \rangle \, d\,\mathrm{vol}_h$$

$$\nabla^2 \mathscr{Y}_N(A\,; A_0)[\varphi, \psi] = \int_N (\langle d_A \varphi, d_A \psi \rangle + \langle F_A, [\varphi \wedge \psi] \rangle) \, d\,\mathrm{vol}_h$$

for every $\varphi, \psi \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})$.

Proof. Since $\mathscr{Y}_N(\,\cdot\,; A_0)$ is simply a shift of YM_N by a the constant additive factor $\mathrm{YM}_N(A_0)$, (ii) follows directly from (i). Hence, we turn to show (i). Fix any $A \in (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})$. Then, by direct computation, for every $\varphi \in W^{1,2}(N, T^*N \otimes \mathfrak{g})$ we have

$$\begin{aligned}
\mathrm{YM}_N(A + \varphi) &= \int_N |F_{A+\varphi}|^2 \, d\,\mathrm{vol}_h \\
&= \int_N |F_A + d_A \varphi + \varphi \wedge \varphi|^2 \, d\,\mathrm{vol}_h \\
&= \int_N |F_A|^2 + 2 \int_N \langle F_A, d_A \varphi \rangle \, d\,\mathrm{vol}_h + \int_N \big(|d_A \varphi|^2 + \langle F_A, [\varphi \wedge \varphi[\rangle \big) \, d\,\mathrm{vol}_h \\
&\quad + 2 \int_N \langle d_A \varphi, \varphi \wedge \varphi \rangle \, d\,\mathrm{vol}_h + \int_N |\varphi \wedge \varphi|^2 \, d\,\mathrm{vol}_h \, .
\end{aligned} \tag{2.1}$$

Let now

$$\begin{aligned}
&\nabla \mathrm{YM}_N(A) : (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g}) \to \mathbb{R} \\
&\nabla^2 \mathrm{YM}_N(A) : (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})^2 \to \mathbb{R} \\
&\nabla^3 \mathrm{YM}_N(A) : (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})^3 \to \mathbb{R} \\
&\nabla^4 \mathrm{YM}_N(A) : (W^{1,2} \cap L^4)(N, T^*N \otimes \mathfrak{g})^4 \to \mathbb{R}
\end{aligned}$$

[10] Here and throughout, by d_A we denote the *exterior covariant derivative* with respect to the connection A, given by

$$d_A \alpha := d\alpha + [A \wedge \alpha] = d\alpha + A \wedge \alpha + \alpha \wedge A.$$

Moreover, we will denote by d_A^* the formal L^2-adjoint operator of d_A.

be given by

$$\nabla \operatorname{YM}_N(A)[\varphi] := 2\int_N \langle F_A, d_A\varphi\rangle \, d\operatorname{vol}_h$$

for every $\varphi \in (W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$,

$$\frac{\nabla^2 \operatorname{YM}_N(A)}{2!}[\varphi_1,\varphi_2] := \int_N \big(\langle d_A\varphi_1, d_A\varphi_2\rangle + \langle F_A, [\varphi_1\wedge\varphi_2]\rangle\big)\, d\operatorname{vol}_h$$

for every $\varphi_1,\varphi_2 \in (W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$,

$$\frac{\nabla^3 \operatorname{YM}_N(A)}{3!}[\varphi_1,\varphi_2,\varphi_3] := 2\int_N \langle d_A\varphi_1, \varphi_2\wedge\varphi_3\rangle \, d\operatorname{vol}_h$$

for every $\varphi_1,\varphi_2,\varphi_3 \in (W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$ and

$$\frac{\nabla^4 \operatorname{YM}_N(A)}{4!}[\varphi_1,\varphi_2,\varphi_3,\varphi_4] := \int_N \langle \varphi_1\wedge\varphi_2, \varphi_3\wedge\varphi_4\rangle \, d\operatorname{vol}_h$$

for every $\varphi_1,\varphi_2,\varphi_3,\varphi_4 \in (W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$. Notice that $\nabla^k \operatorname{YM}_N(A)$ is a k-linear and continous operator on $(W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$ for every $k = 1,...,4$. Moreover, by plugging the definitions of the operators $\nabla^k \operatorname{YM}_N(A)$ in (2.1), we get

$$\begin{aligned}\operatorname{YM}_N(A+\varphi) &= \operatorname{YM}_N(A) + \nabla \operatorname{YM}_N(A)[\varphi] + \frac{\nabla^2 \operatorname{YM}_N(A)}{2}[\varphi,\varphi]\\ &\quad + \frac{\nabla^3 \operatorname{YM}_N(A)}{3!}[\varphi,\varphi,\varphi] + \frac{\nabla^4 \operatorname{YM}_N(A)}{4!}[\varphi,\varphi,\varphi,\varphi]\end{aligned}$$

for every $\varphi \in W^{1,2}(N, T^*N\otimes\mathfrak{g})$. The statement follows. □

2.2. Almost YM-energy minimizers and ω-ASD connections.

Definition 2.5 (Almost YM-energy minimizers). Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 2$ and let (N,h) be a smooth n-dimensional Riemannian manifold, possibly with smooth boundary ∂N. We say that $A \in (W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$ is an *almost* YM-*energy minimizer* if there exist $C,\alpha,\rho_0 > 0$ such that for every geodesic open ball $\mathscr{B}\subset N$ of radius $0<\rho<\rho_0$ such that $\overline{\mathscr{B}}\cap\partial N = \emptyset$ we have

$$\operatorname{YM}_N(A\,;\mathscr{B}) \leq \operatorname{YM}_N(\tilde{A}\,;\mathscr{B}) + C\rho^{n-4+\alpha}, \tag{2.2}$$

for every $\tilde{A} \in (W^{1,2}\cap L^4)(\mathscr{B}, T^*\mathscr{B}\otimes\mathfrak{g})$ with $\iota^*_{\partial\mathscr{B}}\tilde{A} = \iota^*_{\partial\mathscr{B}}A$.
If the previous inequality holds with $C=0$, then we say that A is a YM-*energy minimizer*.

Definition 2.6 (ω-ASD connections). Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n\geq 4$ and let (N,h) be a smooth, oriented n-dimensional Riemannian manifold, possibly with smooth boundary ∂N. Let $\omega \in C^\infty(N, \wedge^{n-4}T^*N)$ be a smooth $(n-4)$-form on N with unit comass, i.e. such that

$$\|\omega\|_* := \sup\{\omega_x(e_1,...,e_{n-4}) : x\in N,\, e_1,...,e_{n-4}\in T_xN \text{ with } |e_1\wedge...\wedge e_{n-4}|_h = 1\} = 1.$$

We say that $A \in (W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$ is an ω-*anti-self-dual connection* (or, for short, an ω-*ASD connection*) if A satisfies the following first order system of PDEs

$$*F_A = -F_A\wedge\omega. \tag{2.3}$$

Proposition 2.7. *Let $G, \mathfrak{g}$, (N,h) and ω be as in Definition 2.6. Assume that the $\mathfrak{g}$-valued 1-form $A \in (W^{1,2}\cap L^4)(N, T^*N\otimes\mathfrak{g})$ is an ω-ASD connection. Consider the $(n-4)$-current $T_A \in \mathcal{D}_{n-4}(N)$ on N given by*

$$\langle T_A, \alpha\rangle := \int_N \operatorname{tr}(F_A\wedge F_A)\wedge\alpha \qquad \forall \alpha\in\mathcal{D}^{n-4}(N).$$

Then, T_A is an $(n-4)$-cycle semicalibrated by ω, satisfying

$$\mathbb{M}(T_A \llcorner U) = \langle T_A\llcorner U, \omega\rangle = \mathrm{YM}_N(A\,; U) \tag{2.4}$$

for every $U\subset N$ open set such that $U\cap\partial N=\emptyset$, where $\mathbb{M}$ denotes the mass of the current (see [Sim83b]).

Proof. First, we show that T_A is a cycle. Let $\{A_i\}_{i\in\mathbb{N}}\subset C^\infty(N, T^*N\otimes\mathfrak{g})$ be such that $A_i\to A$ strongly in $(W^{1,2}\cap L^4)(N)$. This implies that

$$\operatorname{tr}(F_{A_i}\wedge F_{A_i})\to\operatorname{tr}(F_A\wedge F_A)$$

strongly in $L^1(N)$. Notice that, by the Bianchi identity

$$d_{A_i}F_{A_i} = dF_{A_i} + F_{A_i}\wedge A_i - A_i\wedge F_{A_i} = 0,$$

we have

$$\begin{aligned} d(\operatorname{tr}(F_{A_i}\wedge F_{A_i})) &= \operatorname{tr}(d(F_{A_i}\wedge F_{A_i})) = \operatorname{tr}(dF_{A_i}\wedge F_{A_i} + F_{A_i}\wedge dF_{A_i})\\ &= \operatorname{tr}((A_i\wedge F_{A_i} - F_{A_i}\wedge A_i)\wedge F_{A_i} + F_{A_i}\wedge(A_i\wedge F_{A_i} - F_{A_i}\wedge A_i))\\ &= \operatorname{tr}(A_i\wedge F_{A_i}\wedge F_{A_i} - F_{A_i}\wedge F_{A_i}\wedge A_i)\\ &= \operatorname{tr}(F_{A_i}\wedge F_{A_i}\wedge A_i) - \operatorname{tr}(A_i\wedge F_{A_i}\wedge F_{A_i}) = 0 \qquad \forall\, i\in\mathbb{N}. \end{aligned}$$

Fix any $\alpha\in\mathcal{D}^{n-5}(N)$. By Stokes theorem, we have

$$\int_N \operatorname{tr}(F_{A_i}\wedge F_{A_i})\wedge d\alpha = (-1)^{n-4}\int_N d(\operatorname{tr}(F_{A_i}\wedge F_{A_i}))\wedge\alpha = 0 \qquad \forall\, i\in\mathbb{N}.$$

Moreover

$$\begin{aligned} &\left|\int_N \operatorname{tr}(F_{A_i}\wedge F_{A_i})\wedge d\alpha - \int_N \operatorname{tr}(F_A\wedge F_A)\wedge d\alpha\right|\\ &\qquad\qquad \le \|d\alpha\|_{L^\infty(N)}\|\operatorname{tr}(F_{A_i}\wedge F_{A_i}) - \operatorname{tr}(F_A\wedge F_A)\|_{L^1(N)} \to 0 \end{aligned}$$

as $i\to+\infty$. Thus, we get that

$$\langle\partial T_A, \alpha\rangle = \int_N \operatorname{tr}(F_A\wedge F_A)\wedge d\alpha = 0.$$

By arbitrariness of $\alpha\in\mathcal{D}^{n-5}(N)$, we conclude that $\partial T_A = 0$.
Now we turn to show that T_A is semicalibrated by ω. First, given any open set $U\subset N$ such that $U\cap\partial N=\emptyset$ we notice that

$$\langle T_A\llcorner U, \omega\rangle = \int_U \operatorname{tr}(F_A\wedge F_A)\wedge\omega = \int_U \operatorname{tr}(F_A\wedge F_A\wedge\omega) = -\int_U \operatorname{tr}(F_A\wedge *F_A) = \mathrm{YM}_N(A\,;U).$$

Moreover, take any $\alpha \in \mathcal{D}^{n-4}(U)$ such that $\|\alpha\|_* \leq 1$ and notice that

$$
\begin{aligned}
|\langle T_A \llcorner U, \alpha\rangle| &\leq \int_U |\mathrm{tr}(F_A \wedge F_A) \wedge \alpha| \leq \|\alpha\|_{L^\infty(N)} \int_U |\mathrm{tr}(F_A \wedge F_A)| \, d\,\mathrm{vol}_h \\
&\leq \int_U |\mathrm{tr}(F_A \wedge F_A)| \, d\,\mathrm{vol}_h \leq \int_U |F_A|^2 \, d\,\mathrm{vol}_h = \mathrm{YM}_N(A\,; U).
\end{aligned}
$$

Hence, we have

$$
\mathbb{M}(T_A \llcorner U) := \sup_{\substack{\alpha \in \mathcal{D}^{n-4}(U) \\ \|\alpha\|_* \leq 1}} |\langle T_A \llcorner U, \alpha\rangle| = \langle T_A \llcorner U, \omega\rangle = \mathrm{YM}_N(A\,; U)
$$

and the statement follows. □

Remark 2.8 (ω-ASD connections are almost YM-energy minimizers). By Proposition 2.7, if A is an ω-ASD connection we immediately know that T_A is an almost mass minimizing cycle in the sense of [DLSS17c, Definition 0.1][11]. This means that there exist $C, \alpha, \rho_0 > 0$ such that for every geodesic open ball $\mathscr{B} \subset N$ of radius $0 < \rho < \rho_0$ and for every $S \in \mathcal{D}_{n-3}(N)$ we have

$$
\mathbb{M}(T_A \llcorner \mathscr{B}) \leq \mathbb{M}((T_A + \partial S) \llcorner \mathscr{B}) + C\rho^{n-4+\alpha}.
$$

By (2.4) we then have

$$
\mathrm{YM}_N(A\,; \mathscr{B}) \leq \mathbb{M}((T_A + \partial S) \llcorner \mathscr{B}) + C\rho^{n-4+\alpha}.
$$

Now let $\tilde{A} \in (W^{1,2} \cap L^4)(\mathscr{B}, T^*\mathscr{B} \otimes \mathfrak{g})$ be such that $\iota^*_{\partial\mathscr{B}} \tilde{A} = \iota^*_{\partial\mathscr{B}} A$. It is not hard to show that there exists $S_{\tilde{A}} \in \mathcal{D}_{n-3}(N)$ such that

$$
T_{\tilde{A}} \llcorner \mathscr{B} = (T_A + \partial S_{\tilde{A}}) \llcorner \mathscr{B}.
$$

We infer that

$$
\mathrm{YM}_N(A\,; \mathscr{B}) \leq \mathbb{M}(T_{\tilde{A}} \llcorner \mathscr{B}) + C\rho^{n-4+\alpha}.
$$

Exactly by the same argument that we have used in Proposition 2.7, we can show that

$$
\mathbb{M}(T_{\tilde{A}} \llcorner \mathscr{B}) \leq \mathrm{YM}_N(\tilde{A}\,; \mathscr{B})
$$

and we conclude that

$$
\mathrm{YM}_N(A\,; \mathscr{B}) \leq \mathrm{YM}_N(\tilde{A}\,; \mathscr{B}) + C\rho^{n-4+\alpha}.
$$

Thus, we have shown that every ω-ASD connection A is an almost YM-energy minimizer.

In the following we will need an almost monotonicity formula for almost YM-energy minimizers on open subsets of $\mathbb{R}^n$ for $n \geq 5$. We will obtain such a formula by essentially following the argument developed in [DLSS17c, Proposition 2.1] for almost minimizers of the area functional. Notice that an analogous monotonicity formula was obtained in [CW22, Theorem 16] for ω-ASD connections. Furthermore, in the case of smooth Yang–Mills connections the same formula is essentially due to Price [Pri83] and adapted by Tian in [Tia00, Theorem 2.1.2 and Remark 3].

[11] For a proof of this fact, see [DLSS17c, Proposition 0.4].

Proposition 2.9 (Almost monotonicty formula). *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 5$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Let $A \in (W^{1,2} \cap L^4)(\Omega, T^*\Omega \otimes \mathfrak{g})$ be an almost* YM*-energy minimizer on Ω. Then, there exist $C, \alpha > 0$ such that for every $0 < \sigma < \rho < \operatorname{dist}(y, \partial\Omega)$ we have*

$$\frac{1}{\rho^{n-4}} \int_{B_\rho(y)} |F_A|^2 \, d\mathcal{L}^n - \frac{1}{\sigma^{n-4}} \int_{B_\sigma(y)} |F_A|^2 \, d\mathcal{L}^n + \rho^\alpha \geq C \int_{B_\rho(y) \smallsetminus B_\sigma(y)} \frac{1}{|\cdot|^{n-4}} |F_A \llcorner \nu_y|^2 \, d\mathcal{L}^n,$$

where we have defined

$$\nu_y := \frac{\cdot - y}{|\cdot - y|} \qquad \text{on } \mathbb{R}^n \smallsetminus \{y\}.$$

Proof. Fix $y \in \Omega$. Notice that for $\mathcal{L}^1$-a.e. $0 < r < \operatorname{dist}(y, \partial\Omega)$ we have

$$\iota^*_{\partial B_r(y)} A \in (W^{1,2} \cap L^4)(\partial B_r(y), \wedge^1 \partial B_r(y) \otimes \mathfrak{g}).$$

Hence, for $\mathcal{L}^1$-a.e. $0 < r < \operatorname{dist}(y, \partial\Omega)$ we have

$$A_r := \left(r \frac{\cdot - y}{|\cdot - y|} \right)^* \iota^*_{\partial B_r(y)} A \in (W^{1,2} \cap L^4)(B_r(y), \wedge^1 B_r(y) \otimes \mathfrak{g}).$$

and $\iota^*_{\partial B_r(y)} A_r = \iota^*_{\partial B_r(y)} A$. Thus, by the almost minimality of A (i.e. by (2.2)) and by the coarea formula, we get

$$\begin{aligned}
\mathrm{YM}_\Omega(A\,; B_r(y)) &\leq \mathrm{YM}_\Omega(A_r\,; B_r(y)) + Cr^{n-4+\alpha} \\
&= \int_{B_r(y)} \left| \left(r \frac{x - y}{|x - y|} \right)^* \iota^*_{\partial B_r(y)} F_A \right|^2 d\mathcal{L}^n(x) + Cr^{n-4+\alpha} \\
&= \int_0^r \int_{\partial B_t(y)} \frac{r^4}{t^4} \left| \iota^*_{\partial B_r(y)} F_A \left(\frac{r}{t}(x - y) \right) \right|^2 d\mathscr{H}^{n-1}(x) + Cr^{n-4+\alpha} \qquad (2.5) \\
&= \left(\frac{1}{r^{n-5}} \int_0^r t^{n-5} \, d\mathcal{L}^1(t) \right) \left(\int_{\partial B_r(y)} \left| \iota^*_{\partial B_r(y)} F_A \right|^2 d\mathscr{H}^{n-1} \right) + Cr^{n-4+\alpha} \\
&= \frac{r}{n-4} \int_{\partial B_r(y)} \left| \iota^*_{\partial B_r(y)} F_A \right|^2 d\mathscr{H}^{n-1} + Cr^{n-4+\alpha}
\end{aligned}$$

for $\mathcal{L}^1$-a.e. $0 < r < \operatorname{dist}(y, \partial\Omega)$ and for some $C, \alpha > 0$. Let $f : (0, \operatorname{dist}(y, \partial\Omega)) \to [0, +\infty)$ be given by

$$f(r) := \mathrm{YM}_\Omega(A\,; B_r(y)) = \int_{B_r(y)} |F_A|^2 \, d\mathcal{L}^n \qquad \forall\, r \in (0, \operatorname{dist}(y, \partial\Omega)).$$

Since f is a non-decreasing function on $(0, \operatorname{dist}(y, \partial\Omega))$, in particular f is a function of bounded variation and its distributional derivative Df is a positive measure on $(0, \operatorname{dist}(y, \partial\Omega))$. By the Radon–Nikodym theorem, we have

$$Df := f' \mathcal{L}^1 + \mu_s,$$

where μ_s denotes the singular part of Df with respect to $\mathcal{L}^1$. Multiplying both sides of (2.5) by $(n-4) r^{3-n}$ and then adding Df / r^{n-4} to both sides of the inequality that we have obtained, we get

$$\begin{aligned}\left(\frac{f'(r)}{r^{n-4}}-\frac{1}{r^{n-4}}\int_{\partial B_r(y)}\left|\iota^*_{\partial B_r(y)}F_A\right|^2 d\mathscr{H}^{n-1}\right)\mathcal{L}^1&\\ \leq \frac{\mu_s}{r^{n-4}}+\left(\frac{f'(r)}{r^{n-4}}-\frac{1}{r^{n-4}}\int_{\partial B_r(y)}\left|\iota^*_{\partial B_r(y)}F_A\right|^2 d\mathscr{H}^{n-1}\right)\mathcal{L}^1&\\ \leq \frac{Df}{r^{n-4}}-(n-4)\frac{f(r)}{r^{n-3}}\mathcal{L}^1+\hat{C}r^{\alpha-1}\mathcal{L}^1&\\ = D\left(\frac{f(r)}{r^{n-4}}\right)+\hat{C}r^{\alpha-1}\mathcal{L}^1&\end{aligned}$$

where the equality is intended in the sense of distributions on $(0,\mathrm{dist}(y,\partial\Omega))$ and we have let $\hat{C}:=C(n-4)$. Now, fix any $0<\sigma<\rho<\mathrm{dist}(y,\partial\Omega)$. Integrating the previous inequality on the interval $[\sigma,\rho)$ we get

$$\int_\sigma^\rho \frac{1}{r^{n-4}}\left(f'(r)-\int_{\partial B_r(y)}\left|\iota^*_{\partial B_r(y)}F_A\right|^2 d\mathscr{H}^{n-1}\right)d\mathcal{L}^1(r)\leq \tilde{C}\left(\frac{f(\rho)}{\rho^{n-4}}-\frac{f(\sigma)}{\sigma^{n-4}}+\rho^\alpha\right) \tag{2.6}$$

with $\tilde{C}:=\max\{1,\hat{C}\}$. Notice that, by the coarea formula, we have

$$\begin{aligned}&\int_\sigma^\rho \frac{1}{r^{n-4}}\left(f'(r)-\int_{\partial B_r(y)}\left|\iota^*_{\partial B_r(y)}F_A\right|^2 d\mathscr{H}^{n-1}\right)d\mathcal{L}^1(r)\\ &=\int_\sigma^\rho \frac{1}{r^{n-4}}\left(\int_{\partial B_r(y)}|F_A|^2\,d\mathscr{H}^{n-1}-\int_{\partial B_r(y)}\left|\iota^*_{\partial B_r(y)}F_A\right|^2 d\mathscr{H}^{n-1}\right)d\mathcal{L}^1(r)\\ &=\int_\sigma^\rho \frac{1}{r^{n-4}}\left(\int_{\partial B_r(y)}\left(|F_A|^2-|\iota^*_{\partial B_r(y)}F_A|^2\right)d\mathscr{H}^{n-1}\right)d\mathcal{L}^1(r)\\ &=\int_\sigma^\rho \frac{1}{r^{n-4}}\left(\int_{\partial B_r(y)}|F_A\llcorner\nu_y|^2\,d\mathscr{H}^{n-1}\right)d\mathcal{L}^1(r)\\ &=\int_{B_\rho(y)\smallsetminus B_\sigma(y)}\frac{1}{|\cdot|^{n-4}}|F_A\llcorner\nu_y|^2\,d\mathcal{L}^n.\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we infer that

$$\int_{B_\rho(y)\smallsetminus B_\sigma(y)}\frac{1}{|\cdot|^{n-4}}|F_A\llcorner\nu_y|^2\,d\mathcal{L}^n\leq \tilde{C}\left(\frac{f(\rho)}{\rho^{n-4}}-\frac{f(\sigma)}{\sigma^{n-4}}+\rho^\alpha\right)$$

for every $0<\sigma<\rho<\mathrm{dist}(y,\partial\Omega)$. The statement follows. □

2.3. **Concentration Set.** We now wish to explain the concentration phenomenon appearing in Theorem 1.1, and for which we have to devise a Luckhaus-type analysis, cf. Section 6. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of smooth Yang–Mills connections on Ω with $\mathrm{YM}(A_i)\leq\Lambda<+\infty$. Then, by [Tia00, Proposition 3.1.2] there exists a subsequence $\{A_{i_j}\}$ converging weakly[12] to an admissible Yang–Mills connection A. To this sequence $\{A_i\}_{i\in\mathbb{N}}$ we can associate the following *concentration set*:

$$\Sigma=\bigcap_{r>0}\left\{x\in\Omega;\ \liminf_{i\to\infty} r^{4-n}\int_{B_r(x)}|F_{A_i}|^2\geq\varepsilon_0\right\},$$

[12]Here and throughout, we will interpret weak convergence of connections in the sense of [Tia00, Section 3.1].

where ε_0 is given by [Tia00, Theorem 2.2.1]. In particular, using measure-theoretic arguments one can then prove the bound $\mathcal{H}^{n-4}(\Sigma) \leq C(\Lambda, \varepsilon_0)$. Consider then the Radon measures $\mu_i = |F_{A_i}|^2 d\mathcal{L}^n$. By taking a subsequence if necessary, we may assume $\mu_i \to \mu$ weakly-* as Radon measures on Ω. Fatou's lemma allows us to write

$$\mu = |F_A|^2 d\mathcal{L}^n + \nu,$$

for some nonnegative Radon measure ν on Ω. In other words, ν measures the defect of strong convergence of the curvatures. We can then write the concentration set as $\Sigma = \operatorname{spt}\nu \cup \operatorname{Sing}(A)$, where $\operatorname{Sing}(A)$ is the singular set of A, i.e. the set of points at which A is not regular. Finally, we have that $\nu \equiv 0$ if and only if $\mathcal{H}^{n-4}(\Sigma) = 0$ if and only if the curvatures converge strongly in L^2. Note that some of this analysis goes through when relaxing the regularity of the connections A_i. The set $\Sigma \setminus \operatorname{Sing}(A)$ is usually referred to as the *blow-up locus*. We refer the reader to [Lin99] for a similar analysis in the setting of harmonic maps.

In the case in which all the elements of the sequence $\{A_i\}_{i\in\mathbb{N}}$ are Hermitian Yang–Mills connections on $B_1(0) \subset \mathbb{C}^n$ endowed with a Kähler form ω, with an isolated singularity, the concentration set Σ is a complex analytic subvariety of $\mathbb{C}^n_*$, and the blow-up locus consists precisely of the closure of the codimension two part of Σ. See [CS20a, CS20b, CS21a, CS21b] for further structural results on the concentration set, and blow-up locus, of Hermitian Yang–Mills connections.

3. The slicing lemma and the Lyapunov–Schmidt reduction

The aim of this section is to set the stage for the proof of Theorem 1.6 by proving the crucial slicing lemma, and recalling the classical Lyapunov-Schmidt reduction and adapting it to our setting.

Definition 3.1. Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$ and let $n \geq 5$. We say that $A \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ is *conical* if

$$\iota^*_{\mathbb{S}^{n-1}} A \in (W^{1,2} \cap L^4)(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$$

and

$$A = \left(\frac{\cdot}{|\cdot|}\right)^* \iota^*_{\mathbb{S}^{n-1}} A.$$

Lemma 3.2 (Slicing lemma). *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$ and let $n \geq 5$. Let $A_0 \in (W^{1,2} \cap L^4)(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ and let $\tilde{A}_0 \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ be the 0-homogeneous extension of A_0 inside $\mathbb{B}^n$, i.e.*

$$\tilde{A}_0 := \left(\frac{\cdot}{|\cdot|}\right)^* A_0.$$

Then, for every $A \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ we have

$$\mathscr{Y}_{\mathbb{B}^n}(A\,;\tilde{A}_0) \leq \int_0^1 \mathscr{Y}_{\mathbb{S}^{n-1}}(\Psi^*_\rho A\,;A_0)\rho^{n-5}\,d\mathcal{L}^1(\rho) + \int_0^1 \int_{\mathbb{S}^{n-1}} |\Psi^*_\rho(F_A \llcorner \nu_0)|^2\,d\mathscr{H}^{n-1}\rho^{n-3}\,d\mathcal{L}^1(\rho),$$

where for every $\rho > 0$ the map $\Psi_\rho : \mathbb{S}^{n-1} \to \partial B_\rho(0)$ is the smooth conformal diffeomorphism given by

$$\Psi_\rho(x) := \rho x \qquad \forall\, x \in \mathbb{S}^{n-1}.$$

Moreover, in case A is conical the above simplifies to

$$\mathscr{Y}_{\mathbb{B}^n}(A\,;\tilde{A}_0)=\frac{1}{n-4}\mathscr{Y}_{\mathbb{S}^{n-1}}(\iota^*_{\mathbb{S}^{n-1}}A\,;A_0).$$

Proof. Notice that, for $\mathcal{L}^1$-a.e. $\rho\in(0,1)$ we have that $F_A\in L^2(\partial B_\rho(0))$ with

$$|F_A|^2=|\iota^*_{\partial B_\rho(0)}F_A|^2+|F_A\llcorner\nu_0|^2 \qquad \text{on } \partial B_\rho(0) \tag{3.1}$$

and

$$|F_{\tilde{A}_0}|^2=|\iota^*_{\partial B_\rho(0)}F_{\tilde{A}_0}|^2+|F_{\tilde{A}_0}\llcorner\nu_0|^2 \qquad \text{on } \partial B_\rho(0) \tag{3.2}$$

where equalities are meant in the sense of L^1-functions on $\partial B_\rho(0)$. Notice that

$$\begin{aligned}\iota^*_{\partial B_\rho(0)}F_{\tilde{A}_0}&=\frac{1}{\rho^2}\iota^*_{\mathbb{S}^{n-1}}F_{\tilde{A}_0}\left(\frac{\cdot}{\rho}\right)=\frac{1}{\rho^2}F_{A_0}\left(\frac{\cdot}{\rho}\right)\\ F_{\tilde{A}_0}\llcorner\nu_0&\equiv 0,\end{aligned}$$

so that (3.2) becomes

$$|F_{\tilde{A}_0}|^2=\frac{1}{\rho^4}\left|F_{A_0}\left(\frac{\cdot}{\rho}\right)\right|^2 \qquad \text{on } \partial B_\rho(0) \tag{3.3}$$

for $\mathcal{L}^1$-a.e. $\rho\in(0,1)$. By (3.1), (3.3) and by the coarea formula, we have

$$\begin{aligned}\mathscr{Y}_{\mathbb{B}^n}(A\,;\tilde{A}_0)&=\mathrm{YM}_{\mathbb{B}^n}(A)-\mathrm{YM}_{\mathbb{B}^n}(\tilde{A}_0)=\int_{\mathbb{B}^n}|F_A|^2\,d\mathcal{L}^n-\int_{\mathbb{B}^n}|F_{\tilde{A}_0}|^2\,d\mathcal{L}^n\\
&=\int_0^1\int_{\partial B_\rho(0)}\left(|F_A|^2-|F_{\tilde{A}_0}|^2\right)d\mathscr{H}^{n-1}\,d\mathcal{L}^1(\rho)\\
&=\int_0^1\int_{\partial B_\rho(0)}\left(|\iota^*_{\partial B_\rho(0)}F_A|^2-\frac{1}{\rho^4}\left|F_{A_0}\left(\frac{\cdot}{\rho}\right)\right|^2\right)d\mathscr{H}^{n-1}\,d\mathcal{L}^1(\rho)\\
&\quad+\int_0^1\int_{\partial B_\rho(0)}|F_A\llcorner\nu_0|^2\,d\mathscr{H}^{n-1}\,d\mathcal{L}^1(\rho)\\
&=\int_0^1\rho^{n-5}\int_{\mathbb{S}^{n-1}}\left(|\rho^2\iota^*_{\partial B_\rho(0)}F_A(\rho\,\cdot)|^2-|F_{A_0}|^2\right)d\mathscr{H}^{n-1}\,d\mathcal{L}^1(\rho)\\
&\quad+\int_0^1\rho^{n-3}\int_{\mathbb{S}^{n-1}}|\rho F_A\llcorner\nu_0(\rho\,\cdot)|^2\,d\mathscr{H}^{n-1}\,d\mathcal{L}^1(\rho)\\
&=\int_0^1\rho^{n-5}\int_{\mathbb{S}^{n-1}}\left(|\Psi_\rho^*F_A|^2-|F_{A_0}|^2\right)d\mathscr{H}^{n-1}\,d\mathcal{L}^1(\rho)\\
&\quad+\int_0^1\rho^{n-3}\int_{\mathbb{S}^{n-1}}|\Psi_\rho^*(F_A\llcorner\nu_0)|^2\,d\mathscr{H}^{n-1}\,d\mathcal{L}^1(\rho)\\
&=\int_0^1\mathscr{Y}_{\mathbb{S}^{n-1}}(\Psi_\rho^*A\,;A_0)\rho^{n-5}\,d\mathcal{L}^1(\rho)+\int_0^1\int_{\mathbb{S}^{n-1}}|\Psi_\rho^*(F_A\llcorner\nu_0)|^2\,d\mathscr{H}^{n-1}\rho^{n-3}\,d\mathcal{L}^1(\rho).\end{aligned}$$

The statement follows. □

Remark 3.3. Let $n\geq 5$ and assume that $A_0\in C^\infty(\mathbb{S}^{n-1}\,;T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$. In order to prove Theorem 1.6 we will need a Lyapunov–Schmidt reduction for the energy discrepancy on the sphere $\mathbb{S}^{n-1}$ around its smooth critical points. Nevertheless, there is a clear obstruction to this end that we need to face. Indeed, the second variation $\nabla^2\mathscr{Y}_{\mathbb{S}^{n-1}}(\,\cdot\,;A_0)$ of $\mathscr{Y}_{\mathbb{S}^{n-1}}(\,\cdot\,;A_0)$ has an infinite dimensional kernel, due to the gauge invariance of $\mathrm{YM}_{\mathbb{S}^{n-1}}$. To address this problem, fix any smooth Yang–Mills connection $A\in C^\infty(\mathbb{S}^{n-1};T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$ and notice that A is a smooth critical point of $\mathscr{Y}_{\mathbb{S}^{n-1}}(\,\cdot\,;A_0)$ as well. Recall from Proposition 2.4 that

$$\nabla^2\mathscr{Y}_{\mathbb{S}^{n-1}}(A\,;A_0)[\varphi,\psi]=\int_{\mathbb{S}^{n-1}}(\langle d_A\varphi,d_A\psi\rangle+\langle F_A,[\varphi\wedge\psi]\rangle)\,d\mathscr{H}^{n-1},$$

for every $\varphi,\psi\in C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$. For every $\varphi,\psi\in W^{1,2}(M,T^*M\otimes\mathfrak{g}_P)$ we can rewrite the previous expression as

$$\nabla^2\mathscr{Y}_{\mathbb{S}^{n-1}}(A\,;A_0)[\varphi,\psi]=2\int_{\mathbb{S}^{n-1}}\langle L_A\varphi,\psi\rangle\,d\mathscr{H}^{n-1},$$

where $L_A:C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})\to C^{0,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$ is given by

$$L_A\varphi:=d_A^*d_A\varphi+*[*F_A\wedge\varphi]\qquad\forall\,\varphi\in C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g}).$$

Fix any smooth reference connection $\tilde{A}\in C^\infty(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$ on $\mathbb{S}^{n-1}$ and notice that

$$\begin{aligned}L_A\varphi&=d_A^*(d\varphi+[A\wedge\varphi])+*[*F_A\wedge\varphi]\\&=d_A^*(d_{\tilde{A}}\varphi+[(A-\tilde{A})\wedge\varphi])+*[*F_A\wedge\varphi]\\&=d_A^*d_{\tilde{A}}\varphi+d_A^*([(A-\tilde{A})\wedge\varphi])+*[*F_A\wedge\varphi]\\&=d_{\tilde{A}}^*d_{\tilde{A}}\varphi-(-1)^{n-2}[(A-\tilde{A})\wedge d_{\tilde{A}}\varphi]+d_A^*([(A-\tilde{A})\wedge\varphi])+*[*F_A\wedge\varphi]\\&=d_{\tilde{A}}^*d_{\tilde{A}}\varphi+T_A\varphi,\end{aligned}\tag{3.4}$$

where $T_A:C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})\to C^{0,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$ is the bounded linear operator given by

$$T_A\varphi:=-(-1)^{n-2}[(A-\tilde{A})\wedge d_{\tilde{A}}\varphi]+d_A^*([(A-\tilde{A})\wedge\varphi])+*[*F_A\wedge\varphi]$$

for every $\varphi\in C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$. Hence, the leading term of L_A is given by $d_{\tilde{A}}^*d_{\tilde{A}}$ which is not an elliptic operator. To fix this issue, we need to eliminate the gauge invariance of the Yang–Mills lagrangian in the following way. Let

$$X:=\{A\in C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1})\text{ s.t. }d_{\tilde{A}}^*A=0\}.$$

Assume that $A\in X$ and notice that A is also a critical point of $\mathscr{Y}_{\mathbb{S}^{n-1}}(\,\cdot\,;A_0)\llcorner X$ and that the second variation $\nabla_X^2\mathscr{Y}_{\mathbb{S}^{n-1}}(A\cdot\,;A_0)$ at A of $\mathscr{Y}_{\mathbb{S}^{n-1}}(\,\cdot\,;A_0)\llcorner X$ is the second order liner elliptic differential operator on $\mathbb{S}^{n-1}$ given by

$$\tilde{L}_A\varphi:=\Delta_{\tilde{A}}\varphi+T_A\varphi$$

for every $\varphi\in C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$. Exploiting this fact, in what follows, we will always consider restrictions of the energy discrepancy to suitable subspaces over which its second variation becomes elliptic.

Lemma 3.4 (Lyapunov–Schmidt reduction for the Yang–Mills functional). *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$ and let $n\geq 5$. Let $\tilde{A}\in C^\infty(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1}\otimes\mathfrak{g})$ be any smooth reference connection on $\mathbb{S}^{n-1}$ and let*

$$X:=\{A\in C^{2,\alpha}(\mathbb{S}^{n-1},T^*\mathbb{S}^{n-1})\ \textit{s.t.}\ d_{\tilde{A}}^*A=0\}.$$

Let $A_0 \in C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ be a smooth Yang–Mills connection such that $A_0 \in X$. We know that

$$K := \ker \nabla_X^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)$$

is a finite-dimensional linear subspace of $C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$[13]. Let $K^\perp$ be the orthogonal complement of K inside $L^2(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$. Denote by P_K and $P_K^\perp$ the L^2-orthogonal linear projection operators on the subspaces K and $K^\perp$ respectively. Then, there exists an open neighborhood $U \subset K$ of 0, and an analytic function $\Upsilon\colon K \to K^\perp$ such that the following facts hold.

(i) $\Upsilon(0) = 0$, *and* $\nabla\Upsilon(0) = 0$*;*
(ii) $P_{K^\perp}(\nabla_X \mathscr{Y}_{\mathbb{S}^{n-1}}(\varphi + \Upsilon(\varphi))) = 0$ *for every* $\varphi \in U$*.*
(iii) $P_K(\nabla_X \mathscr{Y}_{\mathbb{S}^{n-1}}(\varphi + \Upsilon(\varphi))) = \nabla\mathfrak{q}(\varphi)$ *for every* $\varphi \in U$*, where* $\mathfrak{q} : U \to \mathbb{R}$ *is the analytic map on* U *given by*

$$\mathfrak{q}(\varphi) := \varphi + \Upsilon(\varphi) \quad \forall\, \varphi \in U.$$

(iv) There exists a constant $C > 0$ *such that very* $\varphi, \eta \in U$*, we have*

$$\|\nabla\Upsilon(\varphi)[\eta]\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \le C\|\eta\|_{C^{0,\alpha}(\mathbb{S}^{n-1})}.$$

4. Proof of the log-epiperimetric inequality for Yang–Mills connections

As by the assumptions of Theorem 1.6, let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \ge 5$ and let $A_0 \in C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ be a smooth $\mathfrak{g}$-valued 1-form on $\mathbb{S}^{n-1}$. Let $\pi : \mathbb{B}^n \smallsetminus \{0\} \to \mathbb{S}^{n-1}$ be given by

$$\pi(x) := \frac{x}{|x|} \qquad \forall\, x \in \mathbb{B}^n \smallsetminus \{0\}$$

and let $\tilde{A}_0 \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ be the 0-homogeneous extension of A_0 inside $\mathbb{B}^n$, given by

$$\tilde{A}_0 := \pi^* A_0.$$

Let $\eta > 0$. Fix any reference connection $\tilde{A} \in C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ such that $\tilde{A} \ne A_0$ and

$$\|A_0 - \tilde{A}\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} < \eta.$$

By [Weh04, Theorem 8.1 and Remark 3.2-(ii)], if $\eta > 0$ is small enough there exists a gauge transformation $g \in C^\infty(\mathbb{S}^{n-1}, G)$ such that

$$d^*_{\tilde{A}} A_0^g = 0.$$

As before, let $X \subset C^{2,\alpha}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ be given by

$$X := \big\{A \in C^{2,\alpha}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g}) \text{ s.t. } d^*_{\tilde{A}} A = 0\big\}.$$

By Remark 3.3, we know that

$$K := \ker \nabla_X^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)$$

is a finite-dimensional linear subspace of $C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$. As in Lemma 3.4, let $K^\perp$ be the orthogonal complement of K inside $L^2(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$. Let $0 \in U \subset K$ and $\Upsilon : U \to K^\perp$ be given by the Lyapunov–Schmidt reduction (Lemma 3.4) of $\mathscr{Y}_{\mathbb{S}^{n-1}}(\,\cdot\,; A_0)|_X$ at its critical point A_0^g. Denote by P_K and $P_{K^\perp}$ the L^2-orthogonal linear projection operators on the

[13] See Remark 3.3.

subspaces K and $K^{\perp}$ respectively. Fix $\delta > 0$ and assume that $A \in C^{2,\alpha}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ is such that

$$\|A - A_0\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} < \delta.$$

Assuming that $\delta < \eta$, we have

$$\|A - \tilde{A}\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \leq \|A - A_0\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|A_0 - \tilde{A}\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} < 2\eta.$$

Possibly choosing $\eta > 0$ smaller, by [Weh04, Theorem 8.1] there exists $h \in C^{3,\alpha}(\mathbb{S}^{n-1}, G)$ such that

$$d^*_{\tilde{A}} A^h = 0.$$

Let $\varphi_A := A^h - A_0^g \in C^{2,\alpha}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ and notice that

$$d^*_{\tilde{A}} \varphi_A = 0,$$

so that $\varphi_A \in X$. By the properties of the Lyapunov–Schmidt reduction, for $\delta > 0$ sufficiently small we have

$$P_K \varphi_A \in U.$$

Thus, we can write

$$\begin{aligned} \varphi_A &= P_K\varphi_A + P_{K^\perp}\varphi_A \\ &= P_K\varphi_A + \Upsilon(P_K\varphi_A) + (P_{K^\perp}\varphi_A - \Upsilon(P_K\varphi_A)) \\ &= P_K\varphi_A + \Upsilon(P_K\varphi_A) + \varphi_A^{\perp}, \end{aligned}$$

where we have defined

$$\varphi_A^{\perp} := P_{K^\perp}\varphi_A - \Upsilon(P_K\varphi_A) \in K^{\perp}.$$

By Remark 3.3, the second variation $\nabla_X^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)$ is induced by an elliptic operator $\mathcal{L}_{\mathscr{Y}}$ on a compact manifold. Since every elliptic operator on a compact manifold has compact resolvent, by the spectral theory for operators with compact resolvent we know that there exist a countable orthonormal basis $\{\phi_j\}_{j\in\mathbb{N}} \subset C^{\infty}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ of $L^2(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ and countably many real numbers[14] $\{\lambda_j\}_{j\in\mathbb{N}}$ such that

$$\mathcal{L}_{\mathscr{Y}}\phi_j = \lambda_j \phi_j \qquad \forall\, j \in \mathbb{N}.$$

Moreover, every eigenvalue λ_j of $\mathcal{L}_{\mathscr{Y}}$ has finite multiplicity. We let

$$\ell := \dim K < +\infty$$

and we assume that the eigenfunctions ϕ_j are ordered in such a way that the set $\{\phi_1, ..., \phi_\ell\}$ forms an orthonormal basis of K. Define the index sets

$$J_+ := \{j \in \mathbb{N} :\, \lambda_j > 0\} \qquad \text{and} \qquad J_- := \{j \in \mathbb{N} :\, \lambda_j < 0\},$$

and we let $\{a_j\}_{j\in J_- \cup J_+} \subset \mathbb{R}$ and $\{b_1, ..., b_\ell\} \subset \mathbb{R}$ be such that

$$\varphi_A^{\perp} = \sum_{j\in J_-} a_j\phi_j + \sum_{j\in J_+} a_j\phi_j =: \varphi_{A,-}^{\perp} + \varphi_{A,+}^{\perp}, \qquad \text{and} \qquad P_K\varphi_A = \sum_{j=1}^{\ell} b_j\phi_j.$$

[14]This follows from the symmetry of $\nabla_X^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(0\,; A_0)$, which translates in the L^2 self-adjointness of $\mathcal{L}_{\mathscr{Y}}$.

Since $P_K\varphi_A \in U$ and U is an open set, there exists $\xi > 0$ with $B^\ell_\xi(b) \subset \mathbb{R}^\ell$ such that for every $x = (x_1, ..., x_\ell) \in B^\ell_\xi(b)$ we have

$$\sum_{j=1}^{\ell} x_j \phi_j \in U.$$

Let $f : B^\ell_\xi(b) \subset \mathbb{R}^\ell \to \mathbb{R}$ be the real-analytic function given by

$$f(x) := \mathscr{Y}_{\mathbb{S}^{n-1}}\left(\sum_{j=1}^{\ell} x_j\phi_j + \Upsilon\left(\sum_{j=1}^{\ell} x_j\phi_j\right); A_0\right) \qquad \forall\, x \in B^\ell_\xi(b). \tag{4.1}$$

Let $t_0 \in (0,1)$ and let $v : [0, t_0] \to B^\ell_\xi(b)$ be the smooth vector field on $B^\ell_\xi(b)$ solving on $[0, t_0]$ the following normalized gradient flow equation for f with initial condition $b = (b_1, ..., b_\ell) \in \mathbb{R}^\ell$:

$$\begin{aligned} v'(t) &= \begin{cases} -\dfrac{\nabla f(v(t))}{|\nabla f(v(t))|} & \text{if } f(v(t)) > \dfrac{f(b)}{2} \\ 0 & \text{otherwise;} \end{cases} \\ v(0) &= b. \end{aligned}$$

Note that this is a *finite dimensional Yang–Mills heat flow*. Let then $\eta, \eta_+ : [0,1] \to \mathbb{R}$ be the cut-off functions given by

$$\eta(\rho) := \varepsilon_f f(b)^{1-\gamma}\sqrt{n-2}C(1-\rho) \quad \text{and} \quad \eta_+(\rho) := 1 - (1-\rho)\alpha\varepsilon, \tag{4.2}$$

for all $\rho \in [0,1]$, and where $\varepsilon, \varepsilon_f, C, \alpha > 0$ and $\gamma \in [0,1)$ are parameters to be chosen later in the proof. For now we just assume that

$$\varepsilon_f f(b)^{1-\gamma}\sqrt{n}C < t_0$$

so that $0 \le \eta < t_0$. Then, let $\mu : \mathbb{B}^n \smallsetminus \{0\} \to \pi^*T^*\mathbb{S}^{n-1} \otimes \mathfrak{g}$ be given by

$$\mu(x) := \sum_{j=1}^{\ell} v_j(\eta(|x|))\phi_j(\pi(x)) \qquad \forall\, x \in \mathbb{B}^n \smallsetminus \{0\}.$$

Define $\varphi_{\hat{A}} \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ by

$$\varphi_{\hat{A}}(x) := \mu(x)[d\pi(x)] + \Upsilon(\mu(x))[d\pi(x)] + (\pi^*\varphi^\perp_{A,-})(x) + \eta_+(|x|)(\pi^*\varphi^\perp_{A,+})(x) \tag{4.3}$$

for every $x \in \mathbb{B}^n \smallsetminus \{0\}$. Lastly, let $\tilde{h} := \pi^*h \in (W^{2,2} \cap W^{1,4})(\mathbb{B}^n, G)$ and define the competitor $\hat{A} \in (W^{1,2} \cap L^4)(\mathbb{B}^n, T^*\mathbb{B}^n \otimes \mathfrak{g})$ to be

$$\hat{A} := (\pi^*A_0^g + \varphi_{\hat{A}})^{\tilde{h}^{-1}}.$$

Notice then that

$$\iota^*_{\mathbb{S}^{n-1}}\hat{A} = (A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A) + \varphi^\perp_{A,-} + \varphi^\perp_{A,+})^{h^{-1}} = (A_0^g + \varphi_A)^{h^{-1}} = A.$$

By Lemma 3.2, we have the bound

$$\begin{aligned}\mathscr{Y}_{\mathbb{B}^n}(\hat{A}\,;\tilde{A}_0)-&(1-\varepsilon)\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}\,;\tilde{A}_0)\\ &\leq \int_0^1 \Big(\mathscr{Y}_{\mathbb{S}^{n-1}}(\Psi_\rho^*\hat{A}\,;A_0)-(1-\varepsilon)\mathscr{Y}_{\mathbb{S}^{n-1}}(A\,;A_0)\Big)\rho^{n-5}\,d\mathcal{L}^1(\rho)\\ &\quad+\int_0^1\int_{\mathbb{S}^{n-1}}|\Psi_\rho^*(F_{\hat{A}}\llcorner\nu_0)|^2\,d\mathscr{H}^{n-1}\rho^{n-3}\,d\mathcal{L}^1(\rho)\\ &:= \mathrm{I}+\mathrm{II},\end{aligned}\tag{4.4}$$

where we defined

$$\mathrm{I}:=\int_0^1\Big(\mathscr{Y}_{\mathbb{S}^{n-1}}(\Psi_\rho^*\hat{A}\,;A_0)-(1-\varepsilon)\mathscr{Y}_{\mathbb{S}^{n-1}}(A\,;A_0)\Big)\rho^{n-5}\,d\mathcal{L}^1(\rho),$$

as well as

$$\mathrm{II}:=\int_0^1\int_{\mathbb{S}^{n-1}}|\Psi_\rho^*(F_{\hat{A}}\llcorner\nu_0)|^2\,d\mathscr{H}^{n-1}\rho^{n-3}\,d\mathcal{L}^1(\rho).$$

Notice that, since $\pi^*A_0^g$ is a conical connection, we have

$$|\Psi_\rho^*(F_{\hat{A}}\llcorner\nu_0)|=|\Psi_\rho^*(F_{\pi^*A_0^g+\varphi_{\hat{A}}}\llcorner\nu_0)|=|\Psi_\rho^*(F_{\varphi_{\hat{A}}}\llcorner\nu_0)|.\tag{4.5}$$

Moreover, by analogous reasons, we have

$$(\varphi_{\hat{A}}\wedge\varphi_{\hat{A}})\llcorner\nu_0=0$$

which implies that

$$\begin{aligned}|\Psi_\rho^*(F_{\varphi_{\hat{A}}}\llcorner\nu_0)|^2&=|\Psi_\rho^*(d\varphi_{\hat{A}}\llcorner\nu_0)|^2\\ &\leq\hat{C}\big((\eta'(\rho))^2(1+\|\nabla\Upsilon(v)[v']\|^2_{L^\infty(0,t_0)})+(\eta'_+(\rho))^2|\varphi^\perp_{A,+}|^2\big),\end{aligned}\tag{4.6}$$

for some constant $\hat{C}>0$ depending only on A_0. Let $C_F>0$ be the constant given by Lemma 3.4-(iv). By plugging the estimate given by Lemma 3.4-(iv) in (4.6) we get

$$|\Psi_\rho^*(F_{\varphi_{\hat{A}}}\llcorner\nu_0)|^2\leq\hat{C}(1+C_F^2)(\eta'(\rho))^2+\hat{C}(\eta'_+(\rho))^2|\varphi^\perp_{A,+}|^2.\tag{4.7}$$

Combining (4.5) and (4.7) we obtain

$$\begin{aligned}\mathrm{II}&=\int_0^1\int_{\mathbb{S}^{n-1}}|\Psi_\rho^*(F_{\hat{A}}\llcorner\nu_0)|^2\,d\mathscr{H}^{n-1}\rho^{n-3}\,d\mathcal{L}^1(\rho)\\ &=\int_0^1\int_{\mathbb{S}^{n-1}}|\Psi_\rho^*(F_{\varphi_{\hat{A}}}\llcorner\nu_0)|^2\,d\mathscr{H}^{n-1}\rho^{n-3}\,d\mathcal{L}^1(\rho)\\ &\leq\int_0^1\int_{\mathbb{S}^{n-1}}\big(\hat{C}(1+C_F^2)(\eta'(\rho))^2+\hat{C}(\eta'_+(\rho))^2|\varphi^\perp_{A,+}|^2\big)\rho^{n-3}\,d\mathcal{L}^1(\rho)\\ &=\hat{C}(1+C_F^2)\mathscr{H}^{n-1}(\mathbb{S}^{n-1})\int_0^1\varepsilon_f^2f(b)^{2-2\gamma}C^2(n-2)\rho^{n-3}\,d\mathcal{L}^1(\rho)\\ &\quad+\hat{C}\|\varphi^\perp_{A,+}\|^2_{L^2(\mathbb{S}^{n-1})}\int_0^1\varepsilon^2\alpha^2\rho^{n-3}\,d\mathcal{L}^1(\rho)\\ &\leq\tilde{C}\big(\varepsilon_f^2f(b)^{2-2\gamma}+\varepsilon^2\|\varphi^\perp_{A,+}\|^2_{L^2(\mathbb{S}^{n-1})}\big),\end{aligned}\tag{4.8}$$

where we have let

$$\tilde{C} := \hat{C}\left((1+C_F^2)\mathscr{H}^{n-1}(\mathbb{S}^{n-1})C^2 + \frac{\alpha^2}{n-2}\right).$$

We can now turn to estimate the first term I. Notice that we can write

$$\begin{aligned}
&\mathscr{Y}_{\mathbb{S}^{n-1}}(\Psi_\rho^*\hat{A}\,;A_0) - (1-\varepsilon)\mathscr{Y}_{\mathbb{S}^{n-1}}(A\,;A_0)\\
&\qquad = \mathscr{Y}_{\mathbb{S}^{n-1}}(\Psi_\rho^*\hat{A}\,;A_0) - \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \Psi_\rho^*(\mu[d\pi] + \Upsilon(\mu)[d\pi])\,;A_0)\\
&\qquad\quad - (1-\varepsilon)\left(\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \varphi_A\,;A_0) - \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A);A_0)\right)\\
&\qquad\quad + \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \Psi_\rho^*(\mu[d\pi] + \Upsilon(\mu)[d\pi])\,;A_0)\\
&\qquad\quad - (1-\varepsilon)\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A)\,;A_0)\\
&\qquad = \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \Psi_\rho^*\varphi_{\hat{A}}\,;A_0) - \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,))\,;A_0)\\
&\qquad\quad - (1-\varepsilon)\left(\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \varphi_A\,;A_0) - \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A);A_0)\right)\\
&\qquad\quad + \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,))\,;A_0)\\
&\qquad\quad - (1-\varepsilon)\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A)\,;A_0)\\
&\qquad = \mathrm{III} + \mathrm{IV}
\end{aligned}$$

where we defined

$$\begin{aligned}
\mathrm{III} = {}& \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \Psi_\rho^*\varphi_{\hat{A}}\,;A_0) - \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,))\,;A_0)\\
&- (1-\varepsilon)\left(\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \varphi_A\,;A_0) - \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A)\,;A_0)\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{IV} = {}& \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,))\,;A_0)\\
&- (1-\varepsilon)\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A)\,;A_0).
\end{aligned}$$

Letting now

$$\psi_\rho := \Psi_\rho^*\varphi_{\hat{A}} - \mu(\rho\,\cdot\,) - \Upsilon(\mu(\rho\,\cdot\,)) \tag{4.9}$$

and, by Taylor expanding around A_0^g, we deduce

$$\begin{aligned}
\mathrm{III} &= \nabla\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,))\,;A_0)[\psi_\rho]\\
&\quad + \frac{1}{2}\nabla^2\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,) + s_1\psi_\rho)\,;A_0)[\psi_\rho,\psi_\rho]\\
&\quad - (1-\varepsilon)\nabla\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A)\,;A_0)[\varphi_A^\perp]\\
&\quad - \frac{1-\varepsilon}{2}\nabla^2\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A) + s_2\varphi_A^\perp\,;A_0)[\varphi_A^\perp,\varphi_A^\perp]\\
&= \frac{1}{2}\nabla^2\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + \mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,) + s_1\psi_\rho)\,;A_0)[\psi_\rho,\psi_\rho]\\
&\quad - \frac{1-\varepsilon}{2}\nabla^2\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A) + s_2\varphi_A^\perp\,;A_0)[\varphi_A^\perp,\varphi_A^\perp]
\end{aligned}$$

for some $s_1, s_2 \in [0,1]$, where in the second equality we have used that $\psi_\rho, \varphi_A^\perp \in K^\perp$ and Lemma 3.4-(ii). By using the analyticity of $\mathscr{Y}_{\mathbb{S}^{n-1}}(\,\cdot\,;A_0)$ around A_0^g, and in particular the fact that its second variation is locally Lipschitz around A_0^g (it is actually smooth around such point), we get that there exists $L > 0$ such that

$$|\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(\xi)[\zeta,\zeta] - \nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(0)[\zeta,\zeta]| \le L\|\xi\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\|\zeta\|^2_{L^2(\mathbb{S}^{n-1})}$$

for every $\zeta \in C^{2,\alpha}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ and $\xi \in C^{2,\alpha}(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ sufficiently close to A_0^g in the $C^{2,\alpha}$-norm. Thus, we get

$$\begin{aligned}
\mathrm{III} \le\ & \frac{1}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\psi_\rho,\psi_\rho] - \frac{1-\varepsilon}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\varphi_A^\perp,\varphi_A^\perp] \\
& + L\|\mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,)) + s_1\psi_\rho\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\|\psi_\rho\|^2_{L^2(\mathbb{S}^{n-1})} \\
& + L\|P_K\varphi_A + \Upsilon(P_K\varphi_A) + s_2\varphi_A^\perp\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})} \\
\le\ & \frac{1}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\psi_\rho,\psi_\rho] - \frac{1-\varepsilon}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\varphi_A^\perp,\varphi_A^\perp] \\
& + L\|\mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,)) + s_1\psi_\rho\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\|\psi_\rho\|^2_{L^2(\mathbb{S}^{n-1})} \\
& + L\big(\|P_K\varphi_A\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|\varphi_A^\perp\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\big)\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})}.
\end{aligned} \tag{4.10}$$

Notice that, by definition of ψ_ρ, we have

$$\psi_\rho := \Psi_\rho^*\varphi_{\hat{A}} - \mu(\rho\,\cdot\,) - \Upsilon(\mu(\rho\,\cdot\,)) = (\varphi_{A,-}^\perp)(x) + \eta_+(|x|)(\varphi_{A,+}^\perp)(x).$$

Hence,

$$\begin{aligned}
& \frac{1}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\psi_\rho,\psi_\rho] - \frac{1-\varepsilon}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\varphi_A^\perp,\varphi_A^\perp] \\
& \quad = \frac{\varepsilon}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\varphi_{A,-}^\perp,\varphi_{A,-}^\perp] + \frac{\eta_+^2(\rho)-(1-\varepsilon)}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\varphi_{A,+}^\perp,\varphi_{A,+}^\perp]
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
& \|\mu(\rho\,\cdot\,) + \Upsilon(\mu(\rho\,\cdot\,)) + s_1\psi_\rho\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\|\psi_\rho\|^2_{W^{1,2}(\mathbb{S}^{n-1})} \\
& \qquad \le C\big(\|\mu(\rho\,\cdot\,)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|\varphi_A^\perp\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\big)\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})}.
\end{aligned} \tag{4.12}$$

By plugging (4.11) and (4.12) in (4.10) we get

$$\begin{aligned}
\mathrm{III} \le\ & \frac{\varepsilon}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\varphi_{A,-}^\perp,\varphi_{A,-}^\perp] + \frac{\eta_+^2(\rho)-(1-\varepsilon)}{2}\nabla^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,; A_0)[\varphi_{A,+}^\perp,\varphi_{A,+}^\perp] \\
& + C\big(\|\mu(\rho\,\cdot\,)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|P_K\varphi_A\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + 2\|\varphi_A^\perp\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\big)\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})}.
\end{aligned} \tag{4.13}$$

By definition of η_+, cf. (4.2), up to choosing $\alpha > 0$ big enough depending on $n \ge 5$ there exists a constant $c > 0$ depending on n such that

$$\int_0^1 (\eta_+^2(\rho) - (1-\varepsilon))\rho^{n-5}\,d\rho \le -c\,\varepsilon.$$

Consequently, multiplying (4.13) by ρ^{n-5} and integrating it with respect to ρ, we infer

(4.14)
$$\begin{aligned}
&\int_0^1 \mathrm{III}\,\rho^{n-5}\,d\mathcal{L}^1(\rho)\\
&\quad\le \varepsilon \max_{\lambda_j<0}\lambda_j\|\varphi_{A,-}^\perp\|^2_{L^2(\mathbb{S}^{n-1})} - c\,\varepsilon\min_{\lambda_j>0}\lambda_j\|\varphi_{A,+}^\perp\|^2_{L^2(\mathbb{S}^{n-1})}\\
&\qquad + C\big(\|\mu(\rho\,\cdot\,)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|P_K\varphi_A\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|\varphi_A^\perp\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\big)\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})}\\
&\quad\le -\Big(C_{A_0}\varepsilon - C\big(\|\mu(\rho\,\cdot\,)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|P_K\varphi_A\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} + \|\varphi_A^\perp\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\big)\Big)\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})},
\end{aligned}$$

where $C_{A_0} > 0$ is a constant depending only on n and the spectral gap of the second variation, thus implying that it depends on A_0. We remark that here we need $n \ge 5$ to have finiteness of the term $\int_0^1 \rho^{n-5}d\rho$. Notice now that using Stokes' theorem, we infer the following pointwise bound

$$|\mu(\rho\,\cdot\,) - P_K(\varphi_A)| \le \int_0^{\eta(\rho)} |d(\mu(t\,\cdot\,))|\,dt \le |\eta(\rho)| \le C\varepsilon_f f(b)^\gamma,$$

as well as

$$|d\mu(\rho\,\cdot\,)| \le \varepsilon_f f(b)^\gamma,$$

so that choosing ε_f sufficiently small, and combining these estimates with elliptic regularity, we have the estimate

$$\|\mu(\rho\,\cdot\,)\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \le 2\|P_K\varphi_A\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}.$$

Whence, choosing $\delta > 0$ sufficiently small (depending on C_{A_0}) and plugging $\|\varphi_A\|_{C^{2,\alpha}}(\mathbb{S}^{n-1}) < \delta$ in (4.14), we infer

$$\int_0^1 \mathrm{III}\,\rho^{n-5}\,d\rho \le -C_{A_0}\varepsilon\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{4.15}$$

We are now left with estimating IV. To this sake, we record Łojasiewicz's inequality for analytic function in $\mathbb{R}^l$, cf. [Loj65].

Lemma 4.1. *Consider an open set $U \subset \mathbb{R}^l$, and an analytic function $h\colon U \to \mathbb{R}$. For every critical point $x \in U$ of h, there exist a neighborhood V of x, an exponent $\gamma \in (0, 1/2]$, and a constant K such that*

$$|h(x) - h(y)|^{1-\gamma} \le K|\nabla h(y)|,$$

for all $y \in V$.

In particular, we can apply Lemma 4.1 to f defined in (4.1), and infer the existence of a neighborhood V of the origin, constants $K > 0$ and $\gamma \in (0, 1/2]$ depending on A_0 and the dimension n such that $|f(v)|^{1-\gamma} \le K|\nabla f(v)|$, for every $v \in V$. Consequently, if $f(v(s)) > 0$, for $0 < s < t$, we have

$$f(v(t)) - f(v(0)) = f(v(t)) - f(b) = \int_0^t \nabla f(v(\tau))\cdot v'(\tau)\,d\tau = -\int_0^t |\nabla f(v(\tau))|\,d\tau \le 0, \tag{4.16}$$

which in turn implies that the function $t \mapsto f(v(t))$ is non-increasing, so that there exists $\overline{\tau} > 0$ such that $f(v(t)) \ge f(b)/2 > 0$, for $0 \le t \le \overline{\tau}$, and $f(v(t)) \le f(b)/2$ if $t \ge \overline{\tau}$. If $\eta(\rho) \le \overline{\tau}$, we have the following

$$\mathrm{IV} = f(v(\eta(\rho))) - (1-\varepsilon)f(b)$$

$$\begin{aligned}
&\leq -\int_0^{\eta(\rho)} |\nabla f(v(\tau))|\, d\tau + \varepsilon f(b) && \text{from (4.16)}\\
&\leq -K\int_0^{\eta(\rho)} |f(v(\tau))|^{1-\gamma}\, d\tau + \varepsilon f(b) && \text{from Lemma 4.1}\\
&\leq -Kf(v(\eta(\rho)))^{1-\gamma}\eta(\rho) + \varepsilon f(b) && \text{monotonicity of } f\\
&\leq -\frac{K}{2^{1-\gamma}} f(b)^{1-\gamma}\eta(\rho) + \varepsilon f(b) && \text{definition of } \overline{\tau}\\
&\leq -\left(\frac{K}{2}\eta(\rho) - \varepsilon f(b)^{\gamma}\right) f(b)^{1-\gamma}.
\end{aligned}$$

Otherwise, if $\eta(\rho) > \overline{\tau}$, we have

$$\mathrm{IV} = f(v(\eta(\rho))) - (1-\varepsilon)f(b) < -\left(\frac{1}{2} - \varepsilon\right) f(b) < -(\eta(\rho) - \varepsilon f(b)^{\gamma})f(b)^{1-\gamma},$$

where for the last inequality we used the inequality $|\eta| \leq C\varepsilon_f f(b)^{1-\gamma} < 1/2$ which holds as long as $f(b)$ is small enough. By letting

$$\tilde{K} := \min\left\{\frac{K}{2}, 1\right\}$$

we obtain

$$\mathrm{IV} \leq -(\tilde{K}\eta(\rho) - \varepsilon f(b)^{\gamma})f(b)^{1-\gamma},$$

and this concludes the estimate for IV.

We are now able to finish the proof of Theorem 1.6. We have two cases.

(a) First, assume $f(b)^{1/2} < \nu\|\varphi_A^{\perp}\|_{L^2(\mathbb{S}^{n-1})}$, for some universal constant ν depending only on A_0, and potentially the gauge g which in turn depends on A_0, and the dimension n. In this case, let $\varepsilon_f = 0$, so that $\eta \equiv 0$, and $\mathrm{IV} = \varepsilon f(b)$. In particular, from (4.4), (4.8) and (4.15), we deduce

$$\begin{aligned}
&\mathscr{Y}_{\mathbb{B}^n}(\hat{A}\,;\tilde{A}_0) - (1-\varepsilon)\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}\,;\tilde{A}_0)\\
&\qquad \leq -C_{A_0}\varepsilon\|\varphi_A^{\perp}\|^2_{L^2(\mathbb{S}^{n-1})} + \hat{C}\varepsilon^2\|\varphi_{A,+}^{\perp}\|^2_{L^2(\mathbb{S}^{n-1})} + \varepsilon f(b)\\
&\qquad \leq -(C_{A_0} - \nu - \hat{C}\varepsilon)\varepsilon\|\varphi_A^{\perp}\|_{L^2(\mathbb{S}^{n-1})} < 0,
\end{aligned}$$

where the last inequality follows by choosing ε and ν appropriately.

(b) Otherwise, we set $\varepsilon = \varepsilon_f f(b)^{1-\gamma}$ for some ε_f sufficiently small depending only on n and u_0, allowing us to estimate IV as follows:

$$\begin{aligned}
\int_0^1 \mathrm{IV}\,\rho^{n-5}\, d\rho &\leq -f(b)^{1-\gamma}\int_0^1 (\tilde{K}\eta(\rho) - \varepsilon f(b)^{\gamma})\rho^{n-5}\, d\rho\\
&= -\varepsilon_f f(b)^{2-2\gamma}\int_0^1 (\tilde{K}C\sqrt{n}(1-\rho) - f(b)^{\gamma})\rho^{n-5}\, d\rho\\
&\leq -\hat{K}\varepsilon_f f(b)^{2-2\gamma},
\end{aligned}$$

for some constant $\hat{K} > 0$, upon taking $C > 0$ larger if necessary. Then, from this inequality, combined with (4.4), (4.8) and (4.15) we infer

$$\mathscr{Y}_{\mathbb{B}^n}(\hat{A}\,;\tilde{A}_0) - (1-\varepsilon)\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}\,;\tilde{A}_0)$$

$$\leq -C_{A_0}\varepsilon\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})} - \hat{K}\varepsilon_f f(b)^{2-2\gamma} + \hat{C}\left(\varepsilon_f^2 f(b)^{2-2\gamma} + \varepsilon^2\|\varphi_{A,+}^\perp\|^2_{L^2(\mathbb{S}^{n-1})}\right)$$
$$\leq -(C_{A_0}\varepsilon - \varepsilon^2)\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})} - (\hat{K}\varepsilon_f - \hat{C}\varepsilon_f^2)f(b)^{2-2\gamma} < 0,$$

where the last inequality follows by choosing ε_f small enough and the fact that $f(b) > 0$. Thus, we infer

$$\begin{aligned}
\mathscr{Y}_{\mathbb{B}^n}(\hat{A}\,;\tilde{A}_0) &= \frac{1}{n-4}\mathscr{Y}_{\mathbb{S}^{n-1}}(A\,;A_0) \\
&= \frac{1}{n-4}\mathscr{Y}_{\mathbb{S}^{n-1}}(A\,;A_0) \\
&\quad - \frac{1}{n-4}\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A)\,;A_0) \\
&\quad + \frac{1}{n-4}\mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g + P_K\varphi_A + \Upsilon(P_K\varphi_A)\,;A_0) \\
&\leq C_{A_0}\|\varphi_A^\perp\|^2_{L^2(\mathbb{S}^{n-1})} + f(b) \\
&\leq \left(C_{A_0}\nu^{-2} + 1\right) f(b),
\end{aligned}$$

where, in order, we used the slicing Lemma 3.2, a Taylor expansion, and the hypothesis of case (b). Combining the above two inequalities we can conclude the desired log-epiperimetric inequality (upon relabelling the various quantities involved):

$$\mathscr{Y}_{\mathbb{B}^n}(\hat{A}\,;\tilde{A}_0) \leq \left(1 - \varepsilon|\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}\,;\tilde{A}_0)|^\gamma\right)\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}\,;\tilde{A}_0),$$

where ε_f depends only on the dimension n and u_0.

4.1. **The integrable case.** We now specialise the proof of Theorem 1.6 to the case of an integrable cone. We start by recalling from [AS88] this notion. Note that Adams and Simon refer to this property as integrability of the kernel (of the second variation associated to the cone). We will say that $K := \ker \nabla_X^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,;A_0)$ is integrable if for every $v \in K$, there exists a family $\{A_s\}_{s\in(0,1)} \subset C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$ with $A_s \to 0$ in $C^\infty(\mathbb{S}^{n-1}, T^*\mathbb{S}^{n-1} \otimes \mathfrak{g})$, such that $\nabla_X \mathscr{Y}_{\mathbb{S}^{n-1}}(A_s\,;A_0) = 0$ for every $s \in (0,1)$, and $\lim_{s\to 0} A_s/s = v$ in the L^2-sense. In this setting, analyticity of f defined in (4.1) implies the following lemma, whose proof can be found in [AS88, Lemma 1], or [ESV19, Lemma 2.3].

Lemma 4.2. *The integrability condition holds for* $\ker \nabla_X^2 \mathscr{Y}_{\mathbb{S}^{n-1}}(A_0^g\,;A_0)$ *if and only if* $f \equiv f(0)$ *in a neighborhood of* 0.

It is immediate from this lemma that in the proof of the log-epiperimetric inequality we can take $\gamma = 0$, thus obtaining an epiperimetric inequality. The geometric significance of being integrable for a connection A_0 is the following: A_0 has an integrable neighborhood in the moduli space of smooth Yang–Mills connections on the sphere $\mathbb{S}^{n-1}$ with tangent space at A_0 being given by Jacobi fields at it, i.e., solutions of the linearised operator.

5. Proof of the uniqueness of tangent cones with isolated singularities

As by the statement of Theorem 1.1, let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 5$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Let $A \in (W^{1,2} \cap L^4)(\Omega, T^*\Omega \otimes \mathfrak{g})$ be either a YM-energy minimizer or an ω-ASD connection with respect to some smooth semicalibration ω

on Ω. Assume that $\mathscr{H}^{n-4}(\mathrm{Sing}(A)\cap K)<+\infty$ for every compact $K\subset\Omega$. Let $y\in\mathrm{Sing}(A)$. For every $\rho\in(0,\mathrm{dist}(y,\partial\Omega)/2)$, define $A_{y,\rho}\in(W^{1,2}\cap L^4)(B_2(0),\wedge^1 B_2(0)\otimes\mathfrak{g})$ as

$$A_{y,\rho}:=\tau_{y,\rho}^*A,$$

where $\tau_{y,\rho}(x)=\rho x+y$ is the usual rescaling of factor $\rho>0$ centered at y. Let φ be a tangent cone for A at y which is smooth on $B_2(0)\smallsetminus\{0\}$ and such that there exists $\{\rho_i\}_{i\in\mathbb{N}}$ be satisfying $A_{y,\rho_i}\rightharpoonup\varphi$ weakly and $F_{A_{y,\rho_i}}\to F_\varphi$ strongly in L^2 as $i\to+\infty$ (modulo gauge transformations). Let $\varepsilon,\delta>0$ and $\gamma\in[0,1)$ be the constants given by Theorem 1.6 for $A_0=\iota_{\mathbb{S}^{n-1}}^*\varphi$. By the ε-regularity statements in [CW22, Theorem 2] (see also [Tia00] for the energy minimizing case), following the same argument as in [Sim12, Section 3.15] we conclude that the convergence of $\{A_{y,\rho_i}\}_{i\in\mathbb{N}}$ to φ is strong in C^∞ (modulo gauge transformations) on every compact subset K of $B_2(0)\smallsetminus\{0\}$. Hence, there exists $i\in\mathbb{N}$ big enough so that

$$\|A_{y,\rho_i}-\varphi\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\le\|A_{y,\rho_i}-\varphi\|_{C^{2,\alpha}(B_{\frac{3}{2}}(0)\smallsetminus B_{\frac{3}{4}}(0))}<\delta$$

for every $0<\rho\le\rho_i$. Define $\tilde\rho:=\rho_i$. Fix any $k\in\mathbb{N}$. As in [ESV19, Lemma 3.3], we know that for every $\rho\in[\tilde\rho/2^{k+1},\tilde\rho/2^k]$ it holds

$$\|A_{y,\rho}-\varphi\|_{C^{2,\alpha}(\mathbb{S}^{n-1})}\le\|A_{y,\rho}-\varphi\|_{C^{2,\alpha}(B_{\frac{3}{2}}(0)\smallsetminus B_{\frac{3}{4}}(0))}<\delta$$

Thus, by Theorem 1.6, there exists $\hat A_\rho\in(W^{1,2}\cap L^4)(\mathbb{B}^n,T^*\mathbb{B}^n\otimes\mathfrak{g})$ such that

$$\iota_{\mathbb{S}^{n-1}}^*\hat A_\rho=\iota_{\mathbb{S}^{n-1}}^*A_{y,\rho}$$

and

$$\mathscr{Y}_{\mathbb{B}^n}(\hat A_\rho\,;\varphi)\le\big(1-\varepsilon|\mathscr{Y}_{\mathbb{B}^n}(\tilde A_\rho\,;\varphi)|^\gamma\big)\mathscr{Y}_{\mathbb{B}^n}(\tilde A_\rho\,;\varphi).$$

Notice that, since $A_{y,\rho}$ is almost YM-energy minimizing, for some $C_0,\alpha_0>0$ we have

$$\begin{aligned}\Theta(\rho,y;A)-\mathrm{YM}_{\mathbb{B}^n}(\varphi)&=\mathrm{YM}_{\mathbb{B}^n}(A_{y,\rho})-\mathrm{YM}_{\mathbb{B}^n}(\varphi)\\&\le\mathrm{YM}_{\mathbb{B}^n}(\hat A_\rho)-\mathrm{YM}_{\mathbb{B}^n}(\varphi)+C_0\rho^{\alpha_0}\\&=\mathscr{Y}_{\mathbb{B}^n}(\hat A_\rho\,;\varphi)+C_0\rho^{\alpha_0}\\&\le\big(1-\varepsilon|\mathscr{Y}_{\mathbb{B}^n}(\tilde u_\rho\,;\varphi)|^\gamma\big)\mathscr{Y}_{\mathbb{B}^n}(\tilde A_\rho\,;\varphi)+C_0\rho^{\alpha_0}\end{aligned}\tag{5.1}$$

for every $\rho\in(\tilde\rho/2^{k+1},\tilde\rho/2^k)$, where $\tilde A_\rho\in(W^{1,2}\cap L^4)(\mathbb{B}^n,T^*\mathbb{B}^n\otimes\mathfrak{g})$ is the 0-homogeneous extension of $A_{y,\rho}$ inside $\mathbb{B}^n$. Let

$$f(\rho):=\rho^{n-4}\left(\Theta(\rho,y;A)-\mathrm{YM}_{\mathbb{B}^n}(\varphi)\right)=\int_{B_\rho(y)}|F_A|^2\,d\mathcal{L}^n-\mathrm{YM}_{\mathbb{B}^n}(\varphi)\rho^{n-4}\qquad\forall\,\rho\in[0,1).$$

Notice that by Proposition 2.9 we have that $[0,1)\ni\rho\mapsto f(\rho)$ is an (almost) non-decreasing function of ρ. Hence, f is differentiable $\mathcal{L}^1$-a.e. and its distributional derivative is a measure whose absolutely continous part (with respect to $\mathcal{L}^1$) coincides $\mathcal{L}^1$-a.e. with the classical differential and whose singular part is non negative. Thus, we have

$$\begin{aligned}f'(\rho)&\ge\int_{\partial B_\rho(y)}|F_A|^2\,d\mathscr{H}^{n-1}-(n-4)\,\mathrm{YM}_{\mathbb{B}^n}(\varphi)\rho^{n-5}\\&=\rho^{n-1}\int_{\mathbb{S}^{n-1}}|F_A(\rho\,\cdot\,+y)|^2\,d\mathscr{H}^{n-1}(x)-(n-4)\,\mathrm{YM}_{\mathbb{B}^n}(\varphi)\rho^{n-5}\end{aligned}$$

$$
\begin{aligned}
&= \rho^{n-5}\left(\int_{\mathbb{S}^{n-1}} |\rho^2 F_A(\rho\,\cdot\,+y)|^2 \, d\mathscr{H}^{n-1}(x) - (n-4)\,\mathrm{YM}_{\mathbb{B}^n}(\varphi)\right) \\
&= \rho^{n-5}\left(\int_{\mathbb{S}^{n-1}} |F_{A_{y,\rho}}|^2 \, d\mathscr{H}^{n-1}(x) - (n-4)\,\mathrm{YM}_{\mathbb{B}^n}(\varphi)\right) \\
&= \rho^{n-5}(n-4)\big(\mathrm{YM}_{\mathbb{B}^n}(\tilde{A}_\rho) - \mathrm{YM}_{\mathbb{B}^n}(\varphi)\big) \\
&= \rho^{n-5}(n-4)\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}_\rho\,;\varphi) \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0,1),
\end{aligned}
$$

which can be rewritten as

$$
\rho^{n-4}\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}_\rho\,;\varphi) \le \frac{\rho}{n-4} f'(\rho) \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0,1). \tag{5.2}
$$

By plugging (5.2) in (5.1) we get

$$
\begin{aligned}
f(\rho) &= \rho^{n-4}\left(\Theta(\rho, y; A) - \mathrm{YM}_{\mathbb{B}^n}(\varphi)\right) \\
&\le \big(1 - \varepsilon|\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}_\rho\,;\varphi)|^\gamma\big)\rho^{n-4}\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}_\rho\,;\varphi) + C_0\rho^{n-4+\alpha_0} \\
&\le \big(1 - \varepsilon|\mathscr{Y}_{\mathbb{B}^n}(\tilde{A}_\rho\,;\varphi)|^\gamma\big)\frac{\rho}{n-4} f'(\rho) + C_0\rho^{n-4+\alpha_0}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (\tilde{\rho}/2^{k+1}, \tilde{\rho}/2^k).
\end{aligned} \tag{5.3}
$$

Moreover, since $A_{y,\rho}$ is almost YM-energy minimizing, we have

$$
\begin{aligned}
e(\rho) := \frac{f(\rho)}{\rho^{n-4}} &= \Theta(\rho, y; A) - \mathrm{YM}_{\mathbb{B}^n}(\varphi) \\
&= \frac{1}{\rho^{n-4}} \int_{B_\rho(y)} |F_A|^2 \, d\mathcal{L}^n - \mathrm{YM}_{\mathbb{B}^n}(\varphi) \\
&= \mathrm{YM}_{\mathbb{B}^n}(A_{y,\rho}) - \mathrm{YM}_{\mathbb{B}^n}(\varphi) \\
&\le \mathrm{YM}_{\mathbb{B}^n}(\tilde{A}_\rho) - \mathrm{YM}_{\mathbb{B}^n}(\varphi) + C_0\rho^{\alpha_0} \\
&= \mathscr{Y}_{\mathbb{B}^n}(\tilde{A}_\rho\,;\varphi) + C_0\rho^{\alpha_0} \\
&\le \mathscr{Y}_{\mathbb{B}^n}(\tilde{A}_\rho\,;\varphi) + C_0\rho^{\alpha_0}.
\end{aligned} \tag{5.4}
$$

Hence, by combining (5.3) and (5.4) and letting $\tilde{\varepsilon} = \varepsilon/2$ we get

$$
f(\rho) \le \big(1 - \tilde{\varepsilon}|e(\rho) - C_0\rho^{\alpha_0}|^\gamma\big)\frac{\rho}{n-2} f'(\rho) + C_0\rho^{n-4+\alpha_0}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (\tilde{\rho}/2^{k+1}, \tilde{\rho}/2^k).
$$

Arguing as in [ESV19, Section 3.2, Step 1] we get

$$
e(\rho) \le 2\left(-\,\tilde{\varepsilon}C(n,\gamma)\log\left(\frac{\rho}{\tilde{\rho}}\right)\right)^{-\frac{1}{\gamma}} \qquad \forall\, \rho \in [\tilde{\rho}/2^{k+1}, \tilde{\rho}/2^k],
$$

for some constant $C(n,\gamma) > 0$ depending only on n and γ. Since we have chosen $k \in \mathbb{N}$ arbitrarily and for every $\rho \in (0, \tilde{\rho})$ there exists $k \in \mathbb{N}$ such that $\rho \in [\tilde{\rho}/2^{k+1}, \tilde{\rho}/2^k]$, we have established that

$$
\Theta(\rho, y; A) - \Theta(y\,; A) \le e(\rho) \le 2\left(-\,\tilde{\varepsilon}C(n,\gamma)\log\left(\frac{\rho}{\tilde{\rho}}\right)\right)^{-\frac{1}{\gamma}} \qquad \forall\, \rho \in (0, \tilde{\rho}). \tag{5.5}
$$

The uniqueness of tangent map to u at y then follows directly by Proposition A.1 with $\rho_0 := \tilde{\rho}/2$ and

$$\phi(\rho) := 2\Big(-\tilde{\varepsilon}C(n,\gamma)\log\Big(\frac{\rho}{\tilde{\rho}}\Big)\Big)^{-\frac{1}{\gamma}} \qquad \forall\, \rho \in (0, \tilde{\rho}/2).$$

6. Non-concentration cases: a Luckhaus type lemma for connections

In this section we deal with the possibility of concentration of measures. We start by proving the following *Luckhaus type* lemma for Sobolev connections in dimension $n \geq 5$.

Lemma 6.1 (Luckhaus type lemma for connections). *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 5$, $\rho \in (0,1)$ and $\lambda \in (0,1/2)$. Let $A_1 \in W^{1,\frac{n-1}{2}}(\mathbb{S}_\rho, T^*\mathbb{B}_\rho \otimes \mathfrak{g})$, $A_2 \in W^{1,\frac{n-1}{2}}(\mathbb{S}_{(1-\lambda)\rho}, T^*\mathbb{B}_{(1-\lambda)\rho} \otimes \mathfrak{g})$.*
Then, there exists $A_\lambda \in W^{1,\frac{n}{2}}(\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}, T^*(\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}) \otimes \mathfrak{g})$ such that $A_\lambda|_{\mathbb{S}_\rho} = A_1$, $A_\lambda|_{\mathbb{S}_{(1-\lambda)\rho}} = A_2$ and for some constant $K > 0$ we have

$$\int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |F_{A_\lambda}|^2\, d\mathcal{L}^n \leq K\lambda^{\frac{n-4}{n}}\bigg(\int_{\mathbb{S}_\rho}\big(|\nabla A_1|^{\frac{n-1}{2}} + |A_1|^{n-1}\big)\, d\mathscr{H}^{n-1} + \int_{\mathbb{S}_{(1-\lambda)\rho}}\big(|\nabla A_2|^{\frac{n-1}{2}} + |A_2|^{n-1}\big)\, d\mathscr{H}^{n-1}\bigg)^{\frac{4}{n}}.$$

Proof. Recall the continuous Sobolev embeddings

$$W^{1,\frac{n-1}{2}}(\mathbb{S}_\rho) \hookrightarrow W^{1-\frac{2}{n},\frac{n}{2}}(\mathbb{S}_\rho) \qquad \text{and} \qquad W^{1,\frac{n-1}{2}}(\mathbb{S}_{(1-\lambda)\rho}) \hookrightarrow W^{1-\frac{2}{n},\frac{n}{2}}(\mathbb{S}_{(1-\lambda)\rho}). \tag{6.1}$$

Let $\tilde{A}_1 \in W^{1,2}(\mathbb{B}_\rho, T^*\mathbb{B}_\rho \otimes \mathfrak{g})$ and $\tilde{A}_2 \in W^{1,2}(\mathbb{B}_{(1-\lambda)\rho}, T^*\mathbb{B}_{(1-\lambda)\rho} \otimes \mathfrak{g})$ be componentwise harmonic extensions of A_1 and A_2 respectively. By (6.1) and standard elliptic regularity theory, we have

$$\tilde{A}_1 \in W^{1,\frac{n}{2}}(\mathbb{B}_\rho, T^*\mathbb{B}_\rho \otimes \mathfrak{g}) \qquad \text{and} \qquad \tilde{A}_2 \in W^{1,\frac{n}{2}}(\mathbb{B}_{(1-\lambda)\rho}, T^*\mathbb{B}_{(1-\lambda)\rho} \otimes \mathfrak{g})$$

with the following estimates

$$\int_{\mathbb{B}_\rho}\big(|\nabla\tilde{A}_1|^{\frac{n}{2}} + |\tilde{A}_1|^n\big)\, d\mathcal{L}^n \leq K\int_{\mathbb{S}_\rho}\big(|\nabla A_1|^{\frac{n-1}{2}} + |A_1|^{n-1}\big)\, d\mathscr{H}^{n-1}$$
$$\int_{\mathbb{B}_{(1-\lambda)\rho}}\big(|\nabla\tilde{A}_2|^{\frac{n}{2}} + |\tilde{A}_2|^n\big)\, d\mathcal{L}^n \leq K\int_{\mathbb{S}_{(1-\lambda)\rho}}\big(|\nabla A_2|^{\frac{n-1}{2}} + |A_2|^{n-1}\big)\, d\mathscr{H}^{n-1}.$$

Define $\hat{A}_2 \in W^{1,\frac{n}{2}}(\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}, T^*(\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}) \otimes \mathfrak{g})$ by

$$\hat{A}_2 := \bigg(\frac{((1-\lambda)\rho)^2}{|\cdot|^2}\,\cdot\bigg)^*\tilde{A}_2 \qquad \text{on } \mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}.$$

Notice that $\hat{A}_2|_{\mathbb{S}_{(1-\lambda)\rho}} = A_2$ and

$$\int_{\mathbb{B}_\rho \smallsetminus B_{(1-\lambda)\rho}}\big(|\nabla\hat{A}_2|^{\frac{n}{2}} + |\hat{A}_2|^n\big)\, d\mathcal{L}^n \leq \int_{\mathbb{B}_{(1-\lambda)\rho}}\big(|\nabla\tilde{A}_2|^{\frac{n}{2}} + |\tilde{A}_2|^n\big)\, d\mathcal{L}^n \leq K\int_{\mathbb{S}_{(1-\lambda)\rho}}\big(|\nabla A_2|^{\frac{n-1}{2}} + |A_2|^{n-1}\big)\, d\mathscr{H}^{n-1}.$$

Define $\varphi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi_\lambda(x) := \left(\frac{|x|}{\lambda\rho} - \frac{1-\lambda}{\lambda}\right)x \qquad \forall\, x \in \mathbb{R}^n.$$

Define $A_\lambda \in W^{1,\frac{n}{2}}(\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}, T^*(\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}) \otimes \mathfrak{g})$ by

$$A_\lambda := \varphi_\lambda^* \tilde{A}_1 + (\,\cdot\, - \varphi_\lambda)^* \hat{A}_2 \qquad \text{on } \mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}.$$

Notice that since all the norms involved are conformally invariant and φ_λ is a conformal map, we have

$$\begin{aligned}\int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |F_{A_\lambda}|^{\frac{n}{2}}\, d\mathcal{L}^n &\leq K\biggl(\int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |d\tilde{A}_1|^{\frac{n}{2}}\, d\mathcal{L}^n + \int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |d\hat{A}_2|^{\frac{n}{2}}\, d\mathcal{L}^n \\ &\quad + \int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |\tilde{A}_1|^n\, d\mathcal{L}^n + \int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |\hat{A}_2|^n\, d\mathcal{L}^n \\ &\quad + 2\int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |\tilde{A}_1|^{\frac{n}{2}} \cdot |\hat{A}_2|^{\frac{n}{2}}\, d\mathcal{L}^n\biggr) \\ &\leq K\biggl(\int_{\mathbb{S}_\rho} \bigl(|\nabla A_1|^{\frac{n-1}{2}} + |A_1|^{n-1}\bigr)\, d\mathscr{H}^{n-1} \\ &\quad + \int_{\mathbb{S}_{(1-\lambda)\rho}} \bigl(|\nabla A_2|^{\frac{n-1}{2}} + |A_2|^{n-1}\bigr)\, d\mathscr{H}^{n-1}\biggr).\end{aligned}$$

Since by Hölder inequality we have

$$\int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |F_{A_\lambda}|^2\, d\mathcal{L}^n \leq K\lambda^{\frac{n-4}{n}}\left(\int_{\mathbb{B}_\rho \smallsetminus \mathbb{B}_{(1-\lambda)\rho}} |F_{A_\lambda}|^{\frac{n}{2}}\, d\mathcal{L}^n\right)^{\frac{4}{n}},$$

the statement follows. □

The proofs of Theorem 1.3 and Corollary 1.4 follow directly from the following non-concentration lemma for almost YM-energy minimizers in arbitrary dimension.

Lemma 6.2. *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \geq 5$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Let $y \in \Omega$ and let $A \in (W^{1,\frac{n-1}{2}}_{loc} \cap L^{n-1}_{loc})(\Omega \smallsetminus \{y\}, T^*(\Omega \smallsetminus \{y\}) \otimes \mathfrak{g})$ be such that*

$$\int_\Omega |F_A|^{\frac{n-1}{2}}\, d\mathcal{L}^n < +\infty.$$

Assume that A is an almost YM*-minimizing connection with $y \in \mathrm{Sing}(A)$. Assume that $\{\rho_i\}_{i\in\mathbb{N}}$ is such that $\rho_i \to 0$, $A_{y,\rho_i} := \tau_{y,\rho_i}^* A \rightharpoonup \varphi$ weakly and $F_{A_{y,\rho_i}} \to F_\varphi$ weakly in $L^{\frac{n-1}{2}}(\mathbb{B}^n)$. Assume moreover that $\mathrm{Sing}(\varphi) = \{0\}$. Then we have $F_{A_{y,\rho_i}} \to F_\varphi$ strongly in $L^2(\mathbb{B}^n)$ as $i \to +\infty$.*

Proof. First notice that, since for every $i \in \mathbb{N}$ we have $F_{A_{y,\rho_i}} \in L^{\frac{n-1}{2}}(\mathbb{B}^n)$, for every $i \in \mathbb{N}$ there exists $\delta_i \in (0, \frac{1}{2})$ such that $\delta_i \to 0$ as $i \to +\infty$ and

$$\int_{\mathbb{B}^n \smallsetminus \mathbb{B}_{1-\delta_i}} \bigl(|F_{A_{y,\rho_i}}|^{\frac{n-1}{2}} + |F_\varphi|^{\frac{n-1}{2}}\bigr)\, d\mathcal{L}^n < \varepsilon_G,$$

where $\varepsilon_G > 0$ is the constant given by Uhlenbeck's gauge extraction theorem [Uhl82a]. By Fubini's theorem, for every $i \in \mathbb{N}$ there exists $r_i \in (1 - \frac{\delta_i}{2}, 1)$ such that $A_{y,\rho_i} \in W^{1,\frac{n-1}{2}}(\mathbb{S}_{r_i})$ and

$$\int_{\mathbb{S}_{r_i}} |F_{A_{y,\rho_i}}|^{\frac{n-1}{2}}\, d\mathscr{H}^{n-1} < \varepsilon_G,$$

Moreover, for every $i \in \mathbb{N}$ there exists $\lambda_i \in (0, \frac{\delta_i}{2})$ such that

$$\int_{\mathbb{S}_{(1-\lambda_i)r_i}} |F_\varphi|^{\frac{n-1}{2}}\, d\mathscr{H}^{n-1} < \varepsilon_G.$$

Let

$$\varphi_i := \left(\frac{\cdot}{1-\lambda_i}\right)^* \varphi \qquad \text{on } B_{(1-\lambda_i)r_i}.$$

for every $i \in \mathbb{N}$. By Uhlenbeck's Coulomb gauge extraction theorem [Uhl82a], for every $i \in \mathbb{N}$ there exists $g_i \in W^{2,\frac{n-1}{2}}(\mathbb{S}_{r_i}, G)$ and $h_i \in W^{2,\frac{n-1}{2}}(\mathbb{S}_{(1-\lambda_i)r_i}, G)$ such that

$$\int_{\mathbb{S}_{r_i}} \left(|\nabla A_{y,\rho_i}^{g_i}|^{\frac{n-1}{2}} + |A_{y,\rho_i}^{g_i}|^{n-1}\right) d\mathscr{H}^{n-1} \le C_G \int_{\mathbb{S}_{r_i}} |F_{A_{y,\rho_i}}|^{\frac{n-1}{2}}\, d\mathscr{H}^{n-1} < C_G\varepsilon_G$$

and

$$\int_{\mathbb{S}_{r_i}} \left(|\nabla \varphi_i^{h_i}|^{\frac{n-1}{2}} + |\varphi_i^{h_i}|^{n-1}\right) d\mathscr{H}^{n-1} \le C_G \int_{\mathbb{S}_{r_i}} |F_{\varphi_i}|^{\frac{n-1}{2}}\, d\mathscr{H}^{n-1} < C_G\varepsilon_G,$$

where $C_G > 0$ is a constant depending only on G. By Lemma 6.1, for every $i \in \mathbb{N}$ there exists $A_{\lambda_i} \in W^{1,\frac{n}{2}}(\mathbb{B}_{r_i} \smallsetminus \mathbb{B}_{(1-\lambda_i)r_i}, T^*(\mathbb{B}_{r_i} \smallsetminus \mathbb{B}_{(1-\lambda_i)r_i}) \otimes \mathfrak{g})$ such that $A_{\lambda_i}|_{\mathbb{S}_{r_i}} = A_{y,\rho_i}^{g_i}|_{\mathbb{S}_{r_i}}$, $A_{\lambda_i}|_{\mathbb{S}_{(1-\lambda_i)r_i}} = \varphi_i^{h_i}|_{\mathbb{S}_{(1-\lambda_i)r_i}}$ and for some universal $K > 0$ we have

$$\int_{\mathbb{B}_{r_i} \smallsetminus \mathbb{B}_{(1-\lambda_i)r_i}} |F_{A_{\lambda_i}}|^2\, d\mathcal{L}^n \le K C_G \varepsilon_G \lambda_i^{\frac{n-4}{n}}$$

for some constant $K > 0$ depending only on n. Let

$$\tilde{h}_i := \left((1-\lambda_i) r_i \frac{\cdot}{|\cdot|}\right)^* h_i \in W^{2,\frac{n-1}{2}}(\mathbb{B}_{(1-\lambda_i)r_i}, G) \qquad \forall\, i \in \mathbb{N}$$

and define $\tilde{A}_i \in W^{1,\frac{n}{2}}(\mathbb{B}_{r_i}, T^*\mathbb{B}_{r_i} \otimes \mathfrak{g})$ by

$$\tilde{A}_i = \begin{cases} \varphi_i^{\tilde{h}_i} & \text{on } \mathbb{B}_{(1-\lambda_i)r_i} \\ A_{\lambda_i} & \text{on } \mathbb{B}_{r_i} \smallsetminus \mathbb{B}_{(1-\lambda_i)r_i}. \end{cases}$$

Let

$$\tilde{g}_i := \left(r_i \frac{\cdot}{|\cdot|}\right)^* g_i \in W^{2,\frac{n-1}{2}}(\mathbb{B}_{r_i}, G) \qquad \forall\, i \in \mathbb{N}$$

Then, by almost YM-minimality of A, we have

$$\begin{aligned} \int_{\mathbb{B}^n} |F_\varphi|^2\, d\mathcal{L}^n &\le \liminf_{i\to+\infty} \int_{\mathbb{B}^n} |F_{A_{y,\rho_i}}|^2\, d\mathcal{L}^n = \liminf_{i\to+\infty} \int_{\mathbb{B}_{r_i}} \left|F_{A_{y,\rho_i}^{\tilde{g}_i}}\right|^2 d\mathcal{L}^n \\ &\le \liminf_{i\to+\infty} \left(\int_{\mathbb{B}_{r_i}} |F_{\tilde{A}_i}|^2\, d\mathcal{L}^n + C\rho_i^\alpha \right) \end{aligned}$$

$$
\begin{aligned}
&\le \liminf_{i\to+\infty}\left(\int_{\mathbb{B}_{(1-\lambda_i)r_i}} \left|F_{\varphi_i^{\tilde h_i}}\right|^2 d\mathcal{L}^n + \int_{\mathbb{B}_{r_i}\setminus\mathbb{B}_{(1-\lambda_i)r_i}} |F_{A_{\lambda_i}}|^2\, d\mathcal{L}^n\right)\\
&\le \liminf_{i\to+\infty}\left((1-\lambda_i)^{n-4}\int_{\mathbb{B}_{r_i}} |F_\varphi|^2\, d\mathcal{L}^n + KC_G\varepsilon_G\lambda_i^{\frac{n-4}{n}}\right)\\
&= \int_{\mathbb{B}^n} |F_\varphi|^2\, d\mathcal{L}^n.
\end{aligned}
$$

The statement follows. □

Remark 6.3. Let $n \ge 5$, let $\Omega \subset \mathbb{R}^n$ be open and $y \in \Omega$. Let $A \in C^\infty(\Omega\setminus\{y\}, T^*(\Omega\setminus\{y\})\otimes\mathfrak{g})$ is a stationary Yang–Mills connection on $\Omega\setminus\{y\}$. Notice that the uniform pointwise curvature bound

$$
|F_A| \le \frac{C}{|\cdot - y|^2} \qquad \text{on } \Omega\setminus\{y\} \tag{6.2}
$$

for some $C > 0$. Arguing by gluing of gauges on intersecting annuli around y with constant conformal factor as in [Riv20, Proof of Theorem V.6][15], we can show that there exists $\rho > 0$ and $g \in W^{2,\frac{n}{2}}_{loc}(B_\rho(y), G)$ such that A^g satisfies

$$
|\nabla A^g| \le \frac{C}{|\cdot - y|^2} \qquad \text{on } B_\rho(y)\setminus\{y\}
$$

for some $C > 0$. This immediately implies that $A^g \in W^{1,(\frac{n}{2},\infty)}(B_\rho(y))$. Therefore, the assumption in (6.2) used in [Yan03] is strictly stronger than the ones of Lemma 6.2.

Appendix A. A criterion for the uniqueness of tangent cones

The aim of this last section is to prove a standard argument in the literature allowing us to infer uniqueness of tangent cone to an almost YM-energy minimizing Yang–Mills connection A at some point y from a sufficiently fast decay of the energy density $\Theta(\rho, y; A)$ to its limit $\Theta(y; A)$, usually referred to as *Dini continuity*. We reproduce the argument here for the sake of completeness.

Proposition A.1. *Let G be a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let $n \ge 5$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Let $A \in (W^{1,2}\cap L^4)(\Omega, T^*\Omega\otimes\mathfrak{g})$ be an almost* YM*-energy minimizing connection on Ω. Assume that $y \in \operatorname{Sing}(A)$ is an isolated singularity for A and that there exist $\rho_0 \in (0, \operatorname{dist}(y,\partial\Omega))$ and a non-decreasing function $\phi \in (0,\rho_0) \to (0,+\infty)$ such that*

$$
e(\rho) := \Theta(\rho, y; A) - \Theta(y; A) \le \phi(\rho) \qquad \forall\, \rho \in (0,\rho_0) \tag{A.1}
$$

and

$$
\int_0^{\rho_0} \frac{\sqrt{\phi(\rho)}}{\rho}\, d\mathcal{L}^1(\rho) < +\infty. \tag{A.2}
$$

Then, the tangent cone to A at y is unique modulo gauge transformations.

[15] See also the original argument in [Uhl82b].

Proof. First, recall the almost monotonicity formula for almost YM-energy minimizers given by Proposition 2.9: there exist $C, \alpha > 0$ such that for every $0 < \sigma < \rho < \operatorname{dist}(y, \partial\Omega)$ we have

(A.3)
$$\frac{1}{\rho^{n-4}} \int_{B_\rho(y)} |F_A|^2 \, d\mathcal{L}^n - \frac{1}{\sigma^{n-4}} \int_{B_\sigma(y)} |F_A|^2 \, d\mathcal{L}^n + \rho^\alpha \geq C \int_{B_\rho(y) \smallsetminus B_\sigma(y)} \frac{1}{|\cdot|^{n-4}} |F_A \llcorner \nu_y|^2 \, d\mathcal{L}^n,$$

where we have defined

$$\nu_y := \frac{\cdot - y}{|\cdot - y|} \qquad \text{on } \mathbb{R}^n \smallsetminus \{y\}.$$

As $y \in \operatorname{Sing}(A)$ is an isolated singularity for A, there exists $\rho_0 \in (0, \operatorname{dist}(y, \partial\Omega))$ such that A is smooth on $B_{\rho_0}(y) \smallsetminus \{0\}$ Since our result holds modulo gauge transformations, we can always assume that A is in the celebrated Uhlenbeck exponential gauge around y, satisfying

$$A \llcorner \nu_y \equiv 0 \qquad \text{on } B_{\rho_0}(y) \smallsetminus \{y\}. \tag{A.4}$$

By differentiating (A.4), we get

$$\frac{\partial}{\partial \nu_y}(|\cdot - y| A) = |\cdot - y| F_A \llcorner \nu_y \qquad \text{on } B_{\rho_0}(y) \smallsetminus \{y\}. \tag{A.5}$$

Now, let φ_1, φ_2 be any two tangent cones to A at the point y. By definition of tangent cone, there exist sequences $\{\rho_i\}_{i \in \mathbb{N}} \subset (0, \rho_0)$ and $\{\sigma_i\}_{i \in \mathbb{N}} \subset (0, \rho_0)$ such that $\rho_i, \sigma_i \to 0^+$ and

$$A_{y,\rho_i} := \tau^*_{y,\rho_i} A \rightharpoonup \varphi_1 \qquad \text{and} \qquad A_{y,\sigma_i} \rightharpoonup \varphi_2$$

weakly in $W^{1,2}(\mathbb{B}^n)$ as $i \to +\infty$. By the weak continuity of the trace operator, we have

$$A_{y,\rho_i}|_{\mathbb{S}^{n-1}} \rightharpoonup \varphi_1|_{\mathbb{S}^{n-1}} \qquad \text{and} \qquad A_{y,\sigma_i}|_{\mathbb{S}^{n-1}} \rightharpoonup \varphi_2|_{\mathbb{S}^{n-1}}$$

weakly in $L^2(\mathbb{S}^{n-1})$ as $i \to +\infty$. Thus, by (A.5) we have

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} |\varphi_1 - \varphi_2| \, d\mathscr{H}^{n-1} &\leq \liminf_{i \to +\infty} \int_{\mathbb{S}^{n-1}} |A_{y,\rho_i} - A_{y,\sigma_i}| \, d\mathscr{H}^{n-1} \\
&= \liminf_{i \to +\infty} \int_{\mathbb{S}^{n-1}} |\rho_i A(\rho_i x + y) - \sigma_i A(\sigma_i x + y)| \, d\mathscr{H}^{n-1}(x) \\
&= \liminf_{i \to +\infty} \int_{\mathbb{S}^{n-1}} \left| \int_{\sigma_i}^{\rho_i} \frac{\partial}{\partial \rho}(\rho A(\rho x + y)) \, d\mathcal{L}^1(\rho) \right| d\mathscr{H}^{n-1}(x) \\
&= \liminf_{i \to +\infty} \int_{\mathbb{S}^{n-1}} \int_{\sigma_i}^{\rho_i} |(F_A \llcorner \nu_y)(\rho x + y)| \, d\mathcal{L}^1(\rho) \, d\mathscr{H}^{n-1}(x) \\
&= \liminf_{i \to +\infty} \int_{\sigma_i}^{\rho_i} \int_{\mathbb{S}^{n-1}} \rho |(F_A \llcorner \nu_y)(\rho x + y)| \, d\mathscr{H}^{n-1}(x) \, d\mathcal{L}^1(\rho) \\
&= \liminf_{i \to +\infty} \int_{\sigma_i}^{\rho_i} \int_{\partial B_\rho(y)} \frac{1}{\rho^{n-2}} |(F_A \llcorner \nu_y)(z)| \, d\mathscr{H}^{n-1}(z) \, d\mathcal{L}^1(\rho) \\
&= \liminf_{i \to +\infty} \int_{B_{\rho_i}(y) \smallsetminus B_{\sigma_i}(y)} \frac{1}{|\cdot - y|^{n-2}} |F_A \llcorner \nu_y| \, d\mathcal{L}^n. \qquad \text{(A.6)}
\end{aligned}$$

By Hölder inequality, (A.3) and the bound (A.1) in the assumptions, for every $0 < \sigma < \rho < \rho_0$ we have

$$\int_{B_\rho(y) \smallsetminus B_\sigma(y)} \frac{1}{|\cdot - y|^{n-2}} |F_A \llcorner \nu_y| \, d\mathcal{L}^n$$

$$
\begin{aligned}
&\leq \left(\int_{B_\rho(y)\setminus B_\sigma(y)} \frac{1}{|\cdot - y|^{n-4}} |F_A \llcorner \nu_y|^2 \, d\mathcal{L}^n \right)^{\frac{1}{2}} \left(\int_{B_\rho(y)\setminus B_\sigma(y)} \frac{1}{|\cdot - y|^{n}} \, d\mathcal{L}^n \right)^{\frac{1}{2}} \\
&\leq C\big((\log\rho - \log\sigma)(\Theta(\rho, y; A) - \Theta(y; A)) \big)^{\frac{1}{2}} \\
&\leq C\big((\log\rho - \log\sigma)\phi(\rho) \big)^{\frac{1}{2}}.
\end{aligned}
$$

Fix any $0 < \sigma < \rho < \rho_0$ and let $k \in \mathbb{N}$ be such that $\rho/2^k \leq \sigma$. From the previous estimate, for every $i \in \mathbb{N}$ we get

$$
\int_{B_{\rho/2^i}(y)\setminus B_{\rho/2^{i+1}}(y)} \frac{1}{|\cdot - y|^{n-1}} |F_A \llcorner \nu_y| \, d\mathcal{L}^n \leq \tilde{C} \sqrt{\phi\Big(\frac{\rho}{2^i}\Big)}
$$

Then we have

$$
\begin{aligned}
\int_{B_\rho(y)\setminus B_\sigma(y)} \frac{1}{|\cdot - y|^{n-2}} |F_A \llcorner \nu_y| \, d\mathcal{L}^n &\leq \int_{B_\rho(y)\setminus B_{\rho/2^k}(y)} \frac{1}{|\cdot - y|^{n-2}} |F_A \llcorner \nu_y| \, d\mathcal{L}^n \\
&= \sum_{i=0}^{k-1} \int_{B_{\rho/2^i}(y)\setminus B_{\rho/2^{i+1}}(y)} \frac{1}{|\cdot - y|^{n-2}} |F_A \llcorner \nu_y| \, d\mathcal{L}^n \\
&\leq \sum_{i=0}^{n-1} \sqrt{\phi\Big(\frac{\rho}{2^i}\Big)} = \sum_{i=0}^{n-1} \sqrt{\phi\Big(\frac{\rho}{2^i}\Big)} \frac{2^i}{\rho} \frac{\rho}{2^i} \\
&\leq \int_0^\rho \frac{\sqrt{\phi(t)}}{t} \, d\mathcal{L}^1(t).
\end{aligned} \tag{A.7}
$$

By the assumption (A.2) and (A.6) we then get

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}} |\varphi_1 - \varphi_2| \, d\mathscr{H}^{n-1} &\leq \liminf_{i\to+\infty} \int_{B_{\rho_i}(y)\setminus B_{\sigma_i}(y)} \frac{1}{|\cdot - y|^{n-2}} |F_A \llcorner \nu_y| \, d\mathcal{L}^n \\
&\leq \liminf_{i\to+\infty} \int_0^\rho \frac{\sqrt{\phi(t)}}{t} \, d\mathcal{L}^1(t) = 0.
\end{aligned}
$$

Hence, we have $\varphi_1|_{\mathbb{S}^{n-1}} = \varphi_2|_{\mathbb{S}^{n-1}}$. Since both φ_1 and φ_2 are conical connections, we conclude that $\varphi_1 = \varphi_2$ and the statement follows. □

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